# A Clifford algebra gauge invariant Lagrangian for gravity. Part1: higher dimensions and reduction to four-dimensional space-time

Jean Pierre Pansart

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J. P. Pansart[1]
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## Introduction.

Gauge fields have the nature of connexions which distinguishes them from matter fields, which in the present note will be spinor fields. A gauge transformation is a change of local reference frame, and the gauge invariance simply says that the description of physical phenomena does not depend on the local reference frame used. Gauge invariance does not say anything on the nature of the space-time $V$ , but invariance with respect to gauge transformations helps to build Lagrangians.

The main motivation of these notes is that the gravitational field should have the same nature as gauge fields. Like in Kaluza-Klein theories, the space-time will be supposed to be an $n+m$ dimensional manifold : $V=V^{n+m}$ locally of the form : $V^n \otimes V^m$ , where $V^m$ is a compact space of small radius of curvature, invariant under the action of a group $G$ . Usually, in this type of theories, the Lagrangian driving the dynamics of $V$ is the Einstein-Hilbert one. We shall have a different point of view and shall consider the gauge fields associated to the elements of a graded Lie algebra built from the Clifford algebra represented by the Dirac matrices of the $n+m$ dimensional pseudo-euclidean space. This will be explained in chapter 3, which can be read first. The Lagrangian will be the usual quadratic Lagrangian of gauge fields. After reduction, and at the 4 dimensional space-time level, the gravity field Lagrangian will contain four terms : the Einstein-Hilbert Lagrangian given by the space-time scalar curvature, a quadratic term similar to the gauge field Lagrangian, a cosmological constant term and a torsion one. Torsion is naturally introduced and can propagate.

The goal of this note is to show that, after integration on $V^m$ , that is to say at the macroscopic level, one obtains the classical Lagrangian of a spinor field with gravitational and gauge fields, and the usual quadratic Lagrangian of gauge fields. Only the gravitational field Lagrangian will be modified as described above. We show, in a separate note, that this modified Lagrangian gives the same dynamics as the Einstein-Hilbert one in weak gravitational field situations. Differences may appear only in the case strong gravitational fields, but we have not yet studied these consequences.

The notations used in this note are presented in appendix B which recalls also some basic geometrical equations. Chapters 1 and 2 study respectively the consequences of the invariance under the action of the group $G$ , and the transformations of the spinors in

[1] Retired from : Commissariat à l'Energie Atomique, CEN Saclay, DSM/ IRFU/SPP 91191 Gif-sur-Yvette, France. Contact : jean-pierre.pansart@orange.fr

$V = V^{n+m}$ . As said above, chapter 3 presents the construction of the basic Lagrangian. Chapter 4 studies the reduction of the Dirac equation in $V^{n+m}$ in order to obtain the « macroscopic » Dirac equation with gauge fields and gravitational field in $V^n$ . Chapter 5 calculates the different components of the curvature tensor and show how the gauge field Lagrangian of chapter 3 is reduced. At last, chapter 6 look at the mass term of the Dirac equation. A few more hypotheses will be made along the the following chapters. Appendix A shows how to build a possible set of Dirac matrices in $V = V^{n+m}$ .

# 1. Hypotheses and first consequences.

## 1.1 Definitions and hypotheses.

As in Kaluza-Klein type theories , we shall assume that the space-time is an $n+m$ dimensional manifold $V = V^{n+m}$ , locally of the form : $V^n \otimes V^m$ .
The space-time coordinates of a point $x$ are labelled with Greek letters $\alpha, \beta, \gamma$ ... : $\{x^\alpha\}$ , $0 \le \alpha, \beta, \gamma, ... < n+m$ . If it is necessary to distinguish the coordinates, the letters : $\mu, \nu, \eta, \rho$ are used if : $0 \le \alpha < n$ , and : $\varphi, \tau, \chi, \psi$ if : $n \le \alpha < n+m$ . Likewise, when tensor components are expressed with respect to local orthonormal frames, we use Latin letters $a, b, c, ...$ , and the indices : $i, j, k, l, m$ if : $0 \le a < n$ , and : $r, s, t, u$ if : $n \le a < n+m$ .
The symbols used in this note are defined in Appendix B, which provides also a very brief summary of the basic geometrical equations and definitions.

As in Kaluza-Klein theories, $V^m$ is assumed to be invariant under the action of a transformation group $G$ whose parameters are called $\{a^x\}$ : $x'^\varphi = f^\varphi(a,x)$ . The group has no action on $V^n$ : $x'^\mu = f^\mu(a,x) = x^\mu$ .

We set :
$$X_x^\alpha = \frac{\partial f^\alpha(a,x)}{\partial a^x}\Big|_{a^x=0} \tag{1.1}$$
therefore, from the hypothesis :
$$X_x^\mu = 0 \tag{1.2}$$

For an infinitésimal transformation :
$$x'^\alpha = x^\alpha + \eta^x X_x^\alpha \tag{1.3}$$
$|\eta^x| \ll 1$ , the transformation of the orthonormal local frame basis vectors is given by:
$$h'^\alpha_a(x') = h^\alpha_a(x') - \left[\eta^x X_x, h_a\right]^\alpha \tag{1.4}$$
This formula is valid for any vector field, not only for the $\{\overrightarrow{h_a}\}$.
$V^m$ can be considered as an hyper-surface embedded in $V = V^{n+m}$ , and one can chose orthonormal local frames such that $\{\overrightarrow{h_r}\}$ , $n \le r < n+m$ , are tangent to this hyper-surface, by setting :
$$h_r^\mu = 0 \tag{1.5}$$
With the condition (1.2) we then have : $h'^\mu_r(x') = h^\mu_r(x')$ , which shows that the vectors $\{\overrightarrow{h_r}\}$ remain tangent to $V^m$ in a transformation (1.3).

We write : $h_a^\alpha = \begin{pmatrix} h_i^\mu & 0 \\ h_i^\varphi & h_r^\varphi \end{pmatrix}$ , $h_\alpha^a = \begin{pmatrix} h_\mu^i & 0 \\ h_\mu^r & h_\varphi^r \end{pmatrix}$ (1.6)

The differential forms describing the neighborhood of the point where the local frame is defined, are therefore :

$$\omega^i = h_\mu^i \, dx^\mu \qquad \text{et :} \qquad \omega^r = h_\mu^r \, dx^\mu + h_\varphi^r \, dx^\varphi \tag{1.7a}$$

With : $h_a^\alpha \, h_\alpha^b = \delta_a^b$ , one has : $h_i^\mu \, h_\mu^j = \delta_i^j$ and : $h_r^\varphi \, h_\varphi^s = \delta_r^s$ (1.7b)

$$h_i^\mu \, h_\mu^r = -h_i^\varphi \, h_\varphi^r \qquad \text{then :} \qquad h_\mu^r = -h_\mu^i \, h_i^\varphi \, h_\varphi^r \tag{1.7c}$$

and the inverse relation : $h_i^\varphi = -h_i^\mu \, h_\mu^r \, h_r^\varphi$

Likewise, with : $h_\alpha^a \, h_a^\beta = \delta_\alpha^\beta$ , one has : $h_\mu^i \, h_i^\nu = \delta_\mu^\nu$ and $h_\varphi^r \, h_r^\tau = \delta_\varphi^\tau$ (1.7d)

At last : $dx^\mu = h_i^\mu \, \omega^i$ , $dx^\varphi = h_i^\varphi \, \omega^i + h_r^\varphi \, \omega^r$ (1.7e)

From (1.6) the space-time volume element is :

$$dV = \det(h_\alpha^a) \, dx^{\alpha_0} \wedge \ldots \wedge dx^{\alpha_{n+m-1}}$$
$$= (\det(h_\mu^i) \, dx^{\mu_0} \wedge \ldots \wedge dx^{\mu_{n-1}}) \wedge (\det(h_\varphi^r) \, dx^{\varphi_0} \wedge \ldots \wedge dx^{\varphi_{m-1}}) = dV^n \wedge dV^m$$

In Kaluza-Klein theories the gauge fields : $W_\nu^x(x^\nu)$ come from the identification :

$$h_\nu^r = h_\varphi^r \, W_\nu^x(x^\nu) \, X_x^\varphi \tag{1.8}$$

Therefore these gauge fields have not directly the nature of a connexion, however they transform as gauge fields under the action of the group $G$ , as it is now recalled. At first order, one has, using (1.4) :

$$h'^\varphi_i(x) = h_i^\varphi(x) - \left[\eta^x X_x , h_i\right]^\varphi$$

and with (1.8) :

$$h_i^\mu \, W'^x_\mu \, X_x^\varphi = h_i^\mu \, W_\mu^x \, X_x^\varphi - \eta^x \, ( \, X_x^\tau \, \partial_\tau (h_i^\nu \, W_\nu^y \, X_y^\varphi) - h_i^\alpha \, \partial_\alpha X_x^\varphi \, ) + X_x^\varphi \, h_i^\alpha \, \partial_\alpha \eta^x$$

$X_x^\varphi$ , which represents the internal action in $V^m$ of infinitesimal elements of $G$ , does not depend on $\{x^\mu\}$ , and if we assume that : $\partial_\tau h_i^\mu = 0$ , we obtain :

$$h_i^\mu \, W'^x_\mu \, X_x^\varphi = h_i^\mu \, W_\mu^x \, X_x^\varphi - \eta^x \, h_i^\nu \, W_\nu^y \left[X_x , X_y\right] - \eta^x \, X_x^\tau \, h_i^\nu \, X_y^\varphi \, \partial_\tau (W_\nu^y) + X_x^\varphi \, h_i^\alpha \, \partial_\alpha \eta^x$$

Then , if the transformation parameters depend only on the point in $V^n$ , $\eta^x = \eta^x(x^\nu)$ :

$$W'^x_\mu = W_\mu^x + \eta^z \, W_\mu^y \, C^x_{.yz} - \eta^y \, X_y^\tau \, \partial_\tau (W_\mu^x) + \partial_\mu \eta^x$$

If : $\partial_\tau (W_\mu^x) = 0$ , and if : $\partial_\tau h_i^\mu = 0$ , the $W_\mu^x$ transforms like an element of the adjoint representation of the Lie algebra of the group $G$ .

## 1.2 First consequences of the invariance hypothesis.

In all what follows, the coefficients $\eta^x$ of the transformations (1.3) are constants, except if explicitly mentioned.

We define : $\left[\overrightarrow{X_x}, \overrightarrow{h_a}\right] = q_{xa.}{}^b \, \overrightarrow{h_b}$ (1.9a)

and : $h'^\alpha_a(x) = h_a^\alpha(x) - \eta^x \, q_{xa.}{}^b(x) \, h_b^\alpha(x)$ (1.9b)

$$\left[X_x, X_y\right] = C^z_{.xy}\, X_z \tag{1.9c}$$

where the coefficients $C^z_{.yx}$ are the structure constants of $G$ .

Using the Jacobi identities, one can get a relation between the $q_{xa.}{}^{b}$ :

$$\left[X_y, \left[X_x, h_a\right]\right] + \left[h_a, \left[X_y, X_x\right]\right] + \left[X_x, \left[h_a, X_y\right]\right] = 0$$

then :

$$q_{yc.}{}^{b}\; q_{xa.}{}^{c} - q_{xc.}{}^{b}\; q_{ya.}{}^{c} = C^z_{.yx}\, q_{za.}{}^{b} + X_x(q_{ya.}{}^{b}) - X_y(q_{xa.}{}^{b}) \tag{1.10}$$

If $G$ was a rigid group of motion, one would have at first order :

$$\overrightarrow{h'}_a . \overrightarrow{h'}_b = \overrightarrow{h_a} . \overrightarrow{h_b} - \eta^x (q_{xa.}{}^{c}\; \eta_{cb} + q_{xb.}{}^{c}\; \eta_{ca})$$

and setting : $q_{xab} = q_{xa.}{}^{c}\; \eta_{cb}$ , this gives the constraint :

$$q_{xab} + q_{xba} = 0 \tag{1.11}$$

Using the above hypotheses, we now look at the consequences for the coefficients $q_{xa.}{}^{b}$ .

With (1.2) and (1.6) :

$$q_{xr.}{}^{s} = (X^\tau_x\, \partial_\tau h^\varphi_r - h^\tau_r\, \partial_\tau X^\varphi_x)\, h^s_\varphi \tag{1.12a}$$

which is a relation on $V^m$ only.

In the same way :

$$q_{xi.}{}^{j} = X^\tau_x\, \partial_\tau h^\mu_i\; h^j_\mu \tag{1.12b}$$

$$q_{xr.}{}^{i} = 0 \tag{1.12c}$$

which shows that, with (1.4), $\overrightarrow{h'}_r$ remains tangent to $V^m$ , in agreement with the hypotheses.

At last, since $X^\varphi_x$ does not depend on $x^\mu$ :

$$q_{xi.}{}^{r} = (X^\tau_x\, \partial_\tau h^\varphi_i - h^\tau_i\, \partial_\tau X^\varphi_x)\, h^r_\varphi + X^\tau_x\, \partial_\tau h^\mu_i\; h^r_\mu \tag{1.12d}$$

If the torsion is not zero, an infinitesimal parallelogram does not close itself. From that property, and from (1.4), the action of a group of rigid motions would give the constraint :

$$X_x(S^a_{.bc}) - S^a_{.be}(x)\, q_{xc.}{}^{e} - S^a_{.ec}(x)\, q_{xb.}{}^{e} + S^e_{.bc}(x)\, q_{xe.}{}^{a} = 0 \tag{1.13}$$

## 1.3 Commutators and connexion .

The relations (1.5) et (1.6) are constraints on the local orthonormal frame basis vectors. These vectors are related to the connexion coefficients through the structure equations:

$$d\omega^a + \omega^a_{.b} \wedge \omega^b = -\, S^a_{.bc}\; \omega^b \wedge \omega^c = \Sigma^a \tag{1.14}$$

In this section we examine the consequences of these constraints on the connexion coefficients. To do that, the connexion coefficients are written using the basis vector commutators. We define :

$$\left[h_a, h_b\right] = C^c_{.ab}\, h_c \tag{1.15a}$$

then :

$$C^c_{.ab} = \Gamma^c_{.ba} - \Gamma^c_{.ab} + 2\, S^c_{.ab} \tag{1.15b}$$

(Although the same notation has been used for the commutator of the vectors $X_x$ , there will be no ambiguity thanks to the index letters). One has also :

$$\Gamma_{cba} - \Gamma_{cab} = \eta_{cd}\, C^d_{.ab} - 2\, S_{cab} \tag{1.15c}$$

$$2\Gamma_{abc} = \eta_{cd}\, C^d_{.ab} - \eta_{bd}\, C^d_{.ca} - \eta_{ad}\, C^d_{.bc} + 2\, \overline{S}_{abc} \tag{1.15d}$$

where :

$$\overline{S}_{abc} = S_{abc} - S_{cab} - S_{bac} = -\overline{S}_{bac}$$

is the contorsion tensor.

The relation (1.15d) is true only for rigid local frames, and therefore valid for the local orthonormal frames used. Since in (1.15d) the torsion terms are clearly separated, calculations can be performed in the case of zero torsion, and the terms corresponding to it added later. For rigid local frames one has : $\Gamma_{bac} = -\Gamma_{abc}$

then (1.15c) is also : $$\Gamma_{acb} - \Gamma_{bca} = \eta_{cd}\, C^{d}_{.ab} - 2\, S_{cab} \qquad (1.15e)$$

At last, with (1.15b) : $$C^{e}_{.ae} = -\Gamma^{e}_{.ae} + 2\, S^{e}_{.ae} \qquad (1.15f)$$

In (1.15d) we set : $$2\overline{\Gamma}_{abc} = \eta_{cd}\, C^{d}_{.ab} - \eta_{bd}\, C^{d}_{.ca} - \eta_{ad}\, C^{d}_{.bc} \qquad (1.15g)$$

which satisfy : $$\overline{\Gamma}_{cba} - \overline{\Gamma}_{cab} = \eta_{cd}\, C^{d}_{.ab} = \overline{\Gamma}_{acb} - \overline{\Gamma}_{bca} \qquad (1.15h)$$

As it was done for the: $q_{xa.}^{\ \ b}$ in (1.12) , one can now look at the commutation coefficients :

$$C^{l}_{.rs} = \left[h_r, h_s\right]^{\alpha} h^{l}_{\alpha} = (h^{\beta}_{r}\, \partial_{\beta} h^{\mu}_{s} - h^{\beta}_{s}\, \partial_{\beta} h^{\mu}_{r})\, h^{l}_{\mu}$$

and with (1.5) : $$C^{l}_{.rs} = 0 \qquad (1.16a)$$

Likewise : $$C^{l}_{.rj} = h^{\varphi}_{r}\, \partial_{\varphi} h^{\mu}_{j}\, h^{l}_{\mu} \qquad (1.16b)$$

then : $$C^{l}_{.rj} = 0 \quad \text{if} \quad \partial_{\varphi} h^{\mu}_{i} = 0 \qquad (1.16c)$$

$$C^{t}_{.rs} \text{ is computed on } V^{m} \text{ only} \qquad (1.16d)$$

at last: $$C^{k}_{.ij} \text{ is computed on } V^{n} \text{ only if : } \partial_{\varphi} h^{\mu}_{i} = 0 \qquad (1.16e)$$

From (1.15e) , (1.16c) et (1.16a) , one immediately gets the following symmetry relations :

$$\overline{\Gamma}_{ijr} = \overline{\Gamma}_{rji} \quad \text{if : } \partial_{\varphi} h^{\mu}_{i} = 0 \qquad \text{and : } \qquad \overline{\Gamma}_{sjr} = \overline{\Gamma}_{rjs} \qquad (1.16f)$$

moreover : $$\overline{\Gamma}_{rst} \text{ is calculated on } V^{m} \text{ only} \qquad (1.16g)$$

and : $$\overline{\Gamma}_{ijk} \text{ is calculated on } V^{n} \text{ only , if : } \partial_{\varphi} h^{\mu}_{i} = 0 \qquad (1.16h)$$

The other components are : $$2\overline{\Gamma}_{rsi} = \eta_{rt}\, C^{t}_{.is} - \eta_{st}\, C^{t}_{.ir} \qquad (1.16i)$$

$$2\overline{\Gamma}_{irs} = \eta_{st}\, C^{t}_{.ir} + \eta_{rt}\, C^{t}_{.is} \qquad (1.16j)$$

$$2\overline{\Gamma}_{ijr} = \eta_{rs}\, C^{s}_{.ij} \quad \text{if : } \partial_{\varphi} h^{\mu}_{i} = 0 \qquad (1.16k)$$

Let us go back to the definition of the commutator of the local frame basis vectors:

$$(h^{\alpha}_{a}\, \partial_{\alpha} h^{\beta}_{b} - h^{\alpha}_{b}\, \partial_{\alpha} h^{\beta}_{a}) = C^{c}_{.ab}\, h^{\beta}_{c}$$

or equivalently : $$h^{\alpha}_{a}\, h^{\beta}_{b}\, (-\partial_{\alpha} h^{d}_{\beta} + \partial_{\beta} h^{d}_{\alpha}) = C^{d}_{.ab}$$

and set : $$\overline{F}^{d}_{\alpha\beta} \equiv (\partial_{\alpha} h^{d}_{\beta} - \partial_{\beta} h^{d}_{\alpha}) = -C^{d}_{.ab}\, h^{a}_{\alpha}\, h^{b}_{\beta} \qquad (1.17)$$

One obtains $$\partial_{\tau} h^{r}_{\varphi} - \partial_{\varphi} h^{r}_{\tau} + h^{s}_{\varphi}\, h^{t}_{\tau}\, C^{r}_{.ts} = 0 \qquad (1.18a)$$

$$\partial_{\nu} h^{r}_{\varphi} - \partial_{\varphi} h^{r}_{\nu} + h^{s}_{\nu}\, h^{u}_{\varphi}\, C^{r}_{.su} + h^{i}_{\nu}\, h^{u}_{\varphi}\, C^{r}_{.iu} = 0 \qquad (1.18b)$$

$$C^{r}_{.iu} = h^{\nu}_{i}\, h^{\varphi}_{u}\, \overline{F}^{r}_{\varphi\nu} + h^{\varphi}_{i}\, h^{\tau}_{u}\, \overline{F}^{r}_{\tau\varphi} \qquad (1.18c)$$

$$C^{s}_{.ij} = h_i^{\nu} h_j^{\rho} \overline{F}^{s}_{\rho\nu} + (h_i^{\nu} h_j^{\varphi} - h_i^{\varphi} h_j^{\nu}) \overline{F}^{s}_{\varphi\nu} + h_i^{\omega} h_j^{\varphi} \overline{F}^{s}_{\varphi\omega} \qquad (1.18d)$$

In order to get some simplifications we have used several times the condition :

$$\partial_\varphi h_i^\mu = 0 \qquad (1.19a)$$

It will be assumed in all what follows. It implies :

$$\partial_\varphi h_\rho^j = 0 \qquad (1.19b)$$

One will need the following results :

$$\partial_\alpha h_a^\alpha = -\frac{h_a^\beta \partial_\beta \sqrt{g}}{\sqrt{g}} - C^{e}_{.ae} \qquad (1.20)$$

whatever the torsion is, and

$$\partial_\varphi h_r^\varphi = -\frac{h_r^\varphi \partial_\varphi \sqrt{g_{V^m}}}{\sqrt{g_{V^m}}} - C^{s}_{.rs} \qquad (1.21)$$

where : $g_{V^m}$ is the determinant of the $V^m$ metric tensor.

## Summary of this section.

As in Kaluza-Klein theories, the space-time is assumed to be an $n+m$ dimensional manifold $V = V^{n+m}$ , locally of the form : $V^n \otimes V^m$ . $V^m$ is supposed to be invariant under the action of a group of motion $G$ . The basis vectors of the local orthonormal frames are supposed to satisfy (1.5) , (1.6) : $h_r^\mu = 0 \quad \leftrightarrow \quad h_\varphi^i = 0$ .The consequences are given by the relations (1.16). We also impose the condition (1.19) : $\partial_\varphi h_i^\mu = 0$ .

# 2. Spinors in $V^{n+m}$ , action of the $V^m$ invariance group.

In this chapter we study the action of the invariance group $G$ on the spinors of $V^{n+m}$ . In the following, the $\gamma^a$ matrices are supposed to satisfy the constraints :

$$\gamma^{0\,+} = \gamma^0 \qquad \text{and :} \qquad \gamma^0 \gamma^{a+} \gamma^0 = \gamma^a$$

A construction of the $\gamma^a$ matrices in $V^{n+m}$ is presented in appendix A.

## 2.1 Spinor transformations.

Let : $\psi(x)$ be a spinor field defined with respect to a family of orthonormal frames $\{\overrightarrow{h_a}(x)\}$ . If one performs an infinitesimal transformation (1.3) $f:\ x \to x'$ , $\psi(x)$ remains the same in the transformed frames $\{\overrightarrow{h'_a}(x')\}$ which we call adapted frames. In order to know how the spinors are transformed we have to expressed the adapted frames with respect to the local frame $\{\overrightarrow{h_a}(x')\}$ .

We name : $\psi'$ the spinors defined with respect to the family $\{\overrightarrow{h_a}(x')\}$ and we set :

$$\psi' = \Lambda^{-1}\,\psi \qquad \text{and} \qquad \overrightarrow{h'_a}(x') = A^{-1b}_{\ \ a}\,\overrightarrow{h_b}(x') \qquad (2.1)$$

The infinitesimal transformations (1.3) , with constant coefficients, gives, using (1.9b) :

$$\overrightarrow{h'_a}(x') = A^{-1b}_{\ \ a}\, \overrightarrow{h_b}(x') = \overrightarrow{h_a}(x') - \eta^x\, q_{xa.}^{\ \ \ b}(x)\, \overrightarrow{h_b}(x')$$

And, at first order : $\quad A^{-1b}_{\ \ a} = \delta^b_a - \eta^x\, q_{xa.}^{\ \ \ b} \quad , \quad A^b_a = \delta^b_a + \eta^x\, q_{xa.}^{\ \ \ b}$ (2.2)

Usually the transformation of spinors is performed for transformations which conserve the orthonormality of the frames, such as space-time rotations. Here $G$ is a group of rigid transformations for $V^m$ , not necessarily for $V^{n+m}$ .

We shall require that the various tensors built with the spinors transform like tensors :

For scalars : $\quad \overline{\psi'}\, \psi' = \overline{\psi}\, \psi$

For vectors : $\quad \overline{\psi'}\, \gamma^a\, \psi' = A^{-1a}_{\ \ b}\, \overline{\psi}\, \gamma^b\, \psi$

which give the constraints : $\quad \psi^+\, \Lambda^{-1+}\gamma^0\, \Lambda^{-1}\, \psi = \psi^+\, \gamma^0\, \psi$

therefore : $\quad \Lambda^+\, \gamma^0\, \Lambda = \gamma^0$ (2.3a)

and : $\quad \Lambda^{-1}\, \gamma^b\, \Lambda = A^b_a\, \gamma^a$ (2.3b)

For infinitesimal transformations, we set : $\quad \Lambda = I + \eta^x\, M_x$ (2.4)

and at first order : $\quad \Lambda^{-1} = I - \eta^x\, M_x$

since : $\quad \psi'(x') = \psi'(x) + \eta^x\, X^\alpha_x\, \partial_\alpha \psi' = \Lambda^{-1}\, \psi(x)$

we then have : $\quad \psi'(x) = \psi(x) - \eta^x\, (X_x + M_x)\, \psi = \psi(x) - \eta^x\, T_x\, \psi$ (2.5)

The constraints (2.3a) and (2.3b) become respectively :

$$M_x^{\ +}\, \gamma^0 = -\gamma^0\, M_x \qquad (2.6a)$$

and : $\quad \left[\gamma^a, M_x\right] = q_{xb.}^{\ \ \ a}\, \gamma^b$ (2.6b)

What can be said about $M_x$ ?

Let us assume that one can write : $M_x = \alpha_h\, \gamma^h$ , where $\gamma^h = \gamma^{\alpha_1} \ldots \gamma^{\alpha_h}$ is a product of $h$ Dirac matrices in which all the indices are different. If $h = 0$ , $\gamma^h$ is simply a complex number times the unit matrix. For each sequence with $h$ fixed , the commutator of (2.6b) projects on sequences of type $h \pm 1$ , and since the right member of (2.6b) is of degree 1 : $h = 0$ ou $h = 2$ . Therefore : $M_x = S + \alpha_{cd}\, \gamma^c\, \gamma^d$ where : $c \neq d$ . When used back in (2.6b), one gets : $2\,(\alpha_{ab} - \alpha_{ba}) = q_{xba}$ , which implies : $q_{xab} = -q_{xba}$ , and then $G$ should be a group of rigid motion for $V^{n+m}$ , which is not true.

Despite that, we consider : $\quad \widehat{T_x} = X_x - \dfrac{1}{4}\, q_{xcd}\, \gamma^c\, \gamma^d = X_x + \widehat{M_x}$ (2.7)

even if $G$ is not a group of rigid motion, and calculate : $\left[\widehat{T_x}, \widehat{T_y}\right]$ .

Using the relation (1.10) one obtains : $\quad \left[\widehat{T_x}, \widehat{T_y}\right] = C^z_{.xy}\, \widehat{T_z}$ (2.8)

which shows that $\widehat{T_x}$ satisfy the Lie algebra relations of $G$ , but we have to check that $\widehat{M_x}$ satisfies the constraints (2.6b) . We get : $\quad \left[\gamma^a, \widehat{M_x}\right] = \dfrac{1}{2}\,(q_{xb.}^{\ \ \ a} - q_{x.b}^{\ \ a})\, \gamma^b$ . In order to get rid of the difficulty explained above we shall restrict the indices $c, d$ in (2.7) to those corresponding to the part of the local orthonormal frames tangent to $V^m$ , and we set :

$$\widehat{T_x} = X_x - \frac{1}{4}\, q_{xrs}\, \gamma^r\, \gamma^s = X_x + \widehat{M_x} \qquad (2.9)$$

We can now reconsider the preceding calculations. Using : $q_{xrs} + q_{xsr} = 0$ , one gets :

$$\left[\widehat{M}_x, \widehat{M}_y\right] = \frac{1}{4}(q_{xtr}\, q_{ys.}{}^{t} - q_{ytr}\, q_{xs.}{}^{t})\,\gamma^r \gamma^s$$

Now, with (1.12c), one can use (1.10) to show that (2.8) is satisfied. Recall that, in order to obtain this result, we have used the hypothesis that $G$ is a rigid motion group of $V^m$ .

It remains to check that the conditions (2.6a) an (2.6b) are satisfied. Condition (2.6a) is a direct consequence of (A11) et (A12). For the constraint (2.6b) one must check that :

$$\left[\gamma^a, \widehat{M}_x\right] = \left[\gamma^a, -\frac{1}{4} q_{xrs}\, \gamma^r \gamma^s\right] \stackrel{?}{=} q_{xb.}{}^{a}\, \gamma^b$$

If $a = i$ , the commutator is zero, and : $\quad q_{xb.}{}^{i}\, \gamma^b = q_{xt.}{}^{i}\, \gamma^t + q_{xj.}{}^{i}\, \gamma^j$

but from (1.12) : $q_{xt.}{}^{i} = 0$ and $q_{xj.}{}^{i} = 0$ if : $\partial_\varphi h_i^\mu = 0$ . With this last condition, (2.6b) is satisfied for $a = i$ . If $a = t$ , the commutator is :

$$\left[\gamma^t, \frac{1}{4} q_{xrs}\, \gamma^r \gamma^s\right] = \frac{1}{2}(q_{x.s}^{t} - q_{xs.}{}^{t})\,\gamma^s$$

and the constraint is satisfied if : $q_{xrs} = -q_{xsr}$ , which is true if $G$ is a group of rigid motion of $V^m$ .

Finally : $\widehat{T}_x = X_x - \frac{1}{4} q_{xrs}\, \gamma^r \gamma^s = X_x + \widehat{M}_x$ satisfies the Lie algebra relations of $G$ and the constraints (2.6) if $G$ is a group of rigid motion of $V^m$ and if : $\partial_\varphi h_i^\mu = 0$ (2.10)

From (1.9b) and (1.12b) , this last condition means that $h_i^\mu$ is unchanged under the action of $G$ ($q_{xj.}{}^{i} = 0$) , but that $h_i^\varphi$ change ($q_{xi.}{}^{r} \neq 0$) . This fact allows to build gauge fields in Kaluza-Klein theories by defining (1.8).

## 2.2 Hermitic conjugate of $\widehat{T}_x$ .

Let us consider the scalar product : $S = \int_{V^m} \bar{\varphi}\, \psi\, dV$ and let us look at the action of $\widehat{T}_x$ :

$$\bar{\varphi}\,\widehat{T}_x\, \psi \sqrt{g} = \bar{\varphi}\,(X_x^\varphi \partial_\varphi - \frac{1}{4} q_{xrs}\, \gamma^r \gamma^s)\,\psi \sqrt{g} \qquad (r \neq s)$$

$$= -\overline{(X_x^\varphi \partial_\varphi - \frac{1}{4} q_{xrs}\, \gamma^r \gamma^s)\,\varphi}\;\psi \sqrt{g} + \partial_\varphi(\bar{\varphi}\, X_x^\varphi\, \psi \sqrt{g}) - \bar{\varphi}\,\partial_\varphi(X_x^\varphi \sqrt{g})\,\psi$$

For a group of rigid motion which preserves the volume elements, one obtains :

$$\bar{\varphi}\,\widehat{T}_x\, \psi \sqrt{g} = -\overline{\widehat{T}_x \varphi}\;\psi \sqrt{g} + \partial_\varphi(\bar{\varphi}\, X_x^\varphi\, \psi \sqrt{g}) \qquad (2.11)$$

and therefore : $\widehat{T}_x$ is anti-hermitic.

## 2.3 Transformation of the covariant derivative and of the connexion.

We shall first check that the Dirac operator:

$$\gamma^a h_a^\alpha (\partial_\alpha + \Gamma_\alpha - S_\alpha)\psi$$

transforms as expected. Let us replace $\psi$ by $\Lambda\psi'$ , and use (2.3b) :

$$\gamma^a h_a^\alpha (\partial_\alpha + \Gamma_\alpha - S_\alpha)\Lambda\psi' = \gamma^a h_a^\alpha \Lambda (\partial_\alpha + \Lambda^{-1}\Gamma_\alpha \Lambda + \Lambda^{-1}\partial_\alpha \Lambda - S_\alpha)\psi'$$

$$= \Lambda A_b^a \gamma^b h_a^\alpha (\partial_\alpha + \Gamma'_\alpha - S_\alpha)\psi'$$

where we have written : $\Gamma'_\alpha = \Lambda^{-1}\Gamma_\alpha \Lambda + \Lambda^{-1}\partial_\alpha \Lambda$ (2.12)

setting : $\overrightarrow{f_b} = A_b^a \overrightarrow{h_a}$ , one gets the expected transformation law :

$$\gamma^a h_a^\alpha (\partial_\alpha + \Gamma_\alpha - S_\alpha)\psi = \Lambda \ \gamma^b f_b^\alpha (\partial_\alpha + \Gamma'_\alpha - S_\alpha)\psi' \tag{2.13}$$

Now, using (2.9), and at first order : $\Gamma'_\alpha = \Gamma_\alpha + \eta^x [\Gamma_\alpha , M_x] + \partial_\alpha (\eta^x M_x)$

we separate the connexion into 3 contributions : $\Gamma_\alpha = \Gamma_{ij\alpha} \dfrac{\gamma^i \gamma^j}{4} + \Gamma_{ir\alpha} \dfrac{\gamma^i \gamma^r}{2} + \Gamma_{rs\alpha} \dfrac{\gamma^r \gamma^s}{4}$

then : $$\Gamma'_\alpha = \Gamma_\alpha - \eta^x q_{xrs} \Gamma_{ij\alpha} \left[\frac{\gamma^i \gamma^j}{4}, \frac{\gamma^r \gamma^s}{4}\right] - \eta^x q_{xrs} \Gamma_{it\alpha} \left[\frac{\gamma^i \gamma^t}{2}, \frac{\gamma^r \gamma^s}{4}\right]$$

$$-\eta^x q_{xrs} \Gamma_{uv\alpha} \left[\frac{\gamma^u \gamma^v}{4}, \frac{\gamma^r \gamma^s}{4}\right] - \partial_\alpha (\eta^x q_{xrs} \frac{\gamma^r \gamma^s}{4})$$

The first commutator is zero, therefore : $\Gamma'_{ij\alpha} = \Gamma_{ij\alpha}$ (2.14a)

The last two terms are of the form : $S^{-1}\Gamma_\alpha S + S^{-1} dS$ for the group $SO(m)$ whose Lie algebra elements are represented by : $\dfrac{\gamma^r \gamma^s}{2}$ , and therefore represent a gauge transformation in $V^m$ . (2.14b)

At last : $\Gamma'_{is\alpha} \gamma^i \gamma^s = \Gamma_{is\alpha} \gamma^i \gamma^s - \dfrac{1}{2} \eta^x q_{xts} \Gamma^t_{i.\alpha} \gamma^i \gamma^s$ (2.14c)

## 2.4 Hermiticity of the Dirac operator.

We define : $\not{D} = \gamma^a h_a^\alpha (\partial_\alpha + \Gamma_\alpha - S_\alpha)\psi$ (2.15)

and we consider the scalar product : $S = \int_{V^{n+m}} \bar{\varphi} \psi \, dV = \int_{V^{n+m}} \varphi^+ \gamma^0 \psi \, dV$ (2.16)

In order to justify this expression we recall that expressions of the type : $\bar{\varphi}\gamma^{\alpha_1} ... \gamma^{\alpha_h} \psi$ , where all the indices are différents, are rank $h$ tensors with respect to local orthonormal frame rotations, and then : $\bar{\varphi}\psi$ transforms like a scalar for these transformations.

We have : $$\bar{\varphi} \not{D} \psi \sqrt{g} = \partial_\alpha (h_a^\alpha \bar{\varphi} \gamma^a \psi \sqrt{g}) - h_a^\alpha \partial_\alpha \varphi^+ \gamma^{a+} \gamma^0 \psi \sqrt{g}$$

$$+ h_a^\alpha \varphi^+ \gamma^{a+} \gamma^0 (\Gamma_{cd\alpha} \frac{\gamma^c \gamma^d}{4} - S_\alpha)\psi \sqrt{g} - \partial_\alpha (h_a^\alpha \sqrt{g}) \bar{\varphi} \gamma^a \psi$$

the third term on the right is ( with $c \neq d$ ) :

$$h_a^\alpha \varphi^+ \gamma^{a+} (\Gamma_{cd\alpha} \frac{\gamma^{c+} \gamma^{d+}}{4} - S_\alpha) \gamma^0 \psi \sqrt{g} = \left( h_a^\alpha (\Gamma_{cd\alpha} \frac{\gamma^d \gamma^c}{4} - S_\alpha) \gamma^a \varphi \right)^+ \gamma^0 \psi \sqrt{g}$$

$$= \left( -h_a^\alpha \gamma^a (\Gamma_{cd\alpha} \frac{\gamma^c \gamma^d}{4} + S_\alpha) \varphi + \Gamma^a_{.da} \gamma^d \varphi \right)^+ \gamma^0 \psi \sqrt{g}$$

Using the relation : $\Gamma^b_{.ab} = \dfrac{1}{\sqrt{g}} \partial_\alpha (h_a^\alpha \sqrt{g}) + 2 S^\gamma_{.\beta\gamma} h_a^\beta$

we finally obtain : $$\bar{\varphi}\,\not{D}\,\psi\sqrt{g} = -\overline{\not{D}\varphi}\,\psi\sqrt{g} + \partial_\alpha(h_a^\alpha\,\bar{\varphi}\gamma^a\psi\sqrt{g}) \qquad (2.17)$$

## 2.5 The Dirac operator and the commutators.

We shall now write the Dirac operator (2.15) using the local orthonormal frame basis vector commutators. From (1.15d) one has :

$$\gamma^a h_a^\alpha\,\Gamma_\alpha = \Gamma_{cda}\frac{\gamma^a\gamma^c\gamma^d}{4} = \frac{1}{8}(C_{acd} - C_{dac} - C_{cda})\,\gamma^a\gamma^c\gamma^d + \frac{1}{4}(-S_{acd} + S_{dac} + S_{cda})\,\gamma^a\gamma^c\gamma^d$$

which, we recall, is true for rigid frames .

The torsion dependent terms are equal to: $\frac{1}{4}S_{acd}\,(\gamma^a\gamma^c\gamma^d + 6\eta^{ad}\gamma^c)$

In this expression $c \neq d$ but $a$ is free. Separating the various cases, one gets :

$$\frac{1}{4}S_{acd}\,\gamma^a\gamma^c\gamma^d + S^d_{\ .cd}\,\gamma^c \qquad (2.18)$$

where all the indices in the first term are different.

One can do the same thing with the commutation coefficients and finally obtain :

$$\Gamma_{cda}\frac{\gamma^a\gamma^c\gamma^d}{4} - S_a\gamma^a = -\frac{1}{8}C_{acd}\,\gamma^a\gamma^c\gamma^d + \frac{1}{4}S_{acd}\,\gamma^a\gamma^c\gamma^d - \frac{1}{2}C^d_{\ .cd}\,\gamma^c \qquad (2.19)$$

where all the indices in the first two terms on the right are different.

Note that , from (1.16a) , $C^l_{\ .rs} = 0$ , which means that commutators $\left[h_r\,, h_s\right]$ belong to the $V^m$ tangent space, and that, from (1.16d) , these commutators are calculated on $V^m$ only. Then (2.19) is true also on $V^m$ only :

$$\Gamma_{str}\frac{\gamma^r\gamma^s\gamma^t}{4} - \bar{S}_r\gamma^r = -\frac{1}{8}C_{rst}\,\gamma^r\gamma^s\gamma^t + \frac{1}{4}S_{rst}\,\gamma^r\gamma^s\gamma^t - \frac{1}{2}C^s_{\ .rs}\,\gamma^r \qquad (2.20)$$

where : $\bar{S}_r = S^s_{\ .rs} \neq S^c_{\ .rc}$ , and where : $r \neq s \neq t \neq r$ in the two first right member terms .

## Summary.

Under the action of the group of rigid motions $G$ , the spinors are transformed according to :

$$\psi'(x) = \psi(x) - \eta^x\,\widehat{T}_x\,\psi \qquad \text{where :} \qquad \widehat{T}_x = X_x - \frac{1}{4}q_{xrs}\,\gamma^r\gamma^s = X_x + \widehat{M}_x$$

satisfies the $G$ Lie algebra relations and the conditions (2.6a) et (2.6b), if the constraint (1.19a) is satisfied. $\widehat{T}_x$ is anti-hermitic .The covariant derivative of a spinor transforms like a spinor, the Dirac operator (2.15) is anti-hermitic ( $i\not{D}$ is hermitic).

# 3. The gravitational field as a gauge field.

Let us consider the Dirac matrices $\gamma^a$ which represent the basis elements $\{e_a\}$ of the Clifford algebra : $\quad e_a . e_b + e_b . e_a = 2\eta_{ab}$

The commutators : $R^{ab} = \frac{1}{4}\left[\gamma^a, \gamma^b\right]$ represent the generators of the rotation group. They satisfy : $$[R^{ab}, R^{cd}] = -\eta^{ad}R^{cb} - \eta^{ac}R^{bd} - \eta^{bd}R^{ac} - \eta^{bc}R^{da} \qquad (3.1a)$$

and : $$[\gamma^a, \gamma^b] = 4\, R^{ab} \tag{3.1b}$$

$$[\gamma^a, R^{cd}] = \eta^{ac}\gamma^d - \eta^{ad}\gamma^c \tag{3.1c}$$

This set of relations is a graded Lie algebra which satisfies Jacobi's identities. The $\gamma^a$ and $R^{ab}$ matrices are the elements of a representation $\Gamma$ of an algebra defined by the relations (3.1) and named $\Gamma(X_x)$ , where $X_x$ are the basis elements of this algebra. To this algebra we associate a gauge field:

$$W = \alpha\, \omega_{ab}\, R^{ab} + \beta \omega_a\, \gamma^a \tag{3.2}$$

where : $\omega_{ab} = -\omega_{ba}$ and $\omega_a$ are differential forms of degree 1 :

$$\omega_{ab} = \omega_{ab\alpha}\, dx^\alpha \qquad\qquad \omega_a = \eta_{ab}\, \omega^b = \eta_{ab}\, h^b_\alpha\, dx^\alpha \tag{3.3}$$

and : $\alpha$ , $\beta$ are arbitrary constants.

The meaning of these gauge fields is the following : the fields $\omega_{ab}$ are the connexion coefficients defined with respect to a family of local orthonormal frames, an the 1-forms $\omega^a(x) = \eta^{ab}\, \omega_b(x)$ are the coordinates, in the neighborhood of a given point $x$ , with respect to these frames. This can be understood by computing the curvature 2-form (B.8) :

$$G = dW + W \wedge W \tag{3.4}$$

$$G = (\alpha\, d\omega_{ab}\, R^{ab} + \beta\, d\omega_a\, \gamma^a) + (\alpha\, \omega_{cd}\, R^{cd} + \beta \omega_c\, \gamma^c) \wedge (\alpha\, \omega_{ef}\, R^{ef} + \beta \omega_e\, \gamma^e)$$

$$G = \alpha\, d\omega_{ab}\, R^{ab} + \frac{\alpha^2}{2}(\omega_{cd} \wedge \omega_{ef}\, R^{cd}\, R^{ef} + \omega_{ef} \wedge \omega_{cd}\, R^{ef}\, R^{cd})$$

$$+ \beta\, d\omega_a\, \gamma^a + \alpha\beta\, (\omega_{cd} \wedge \omega_e\, R^{cd}\, \gamma^e + \omega_e \wedge \omega_{cd} \wedge \gamma^e R^{cd}) + \beta^2\, \omega_c \wedge \omega_e\, \gamma^c\, \gamma^e$$

and since the gauge fields (3.3) are 1-forms, one has :

$$G = \alpha\, d\omega_{ab}\, R^{ab} + \frac{\alpha^2}{2}\, \omega_{cd} \wedge \omega_{ef}\, [R^{cd}, R^{ef}] + \beta\, d\omega_a\, \gamma^a + \alpha\beta\, \omega_{cd} \wedge \omega_e\, [R^{cd}, \gamma^e] + 2\beta^2\, \omega_a \wedge \omega_b\, R^{ab}$$

With the relations (3.1) this becomes :

$$G = [\alpha\, d\omega_{ab} + 2\alpha^2 \omega_{af} \wedge \omega^f_{.b} + 2\beta^2\, \omega_a \wedge \omega_b] R^{ab} + \beta\, [d\omega_a + 2\alpha\, \omega^e_{a.} \wedge \omega_e]\, \gamma^a \tag{3.5}$$

We can set : $$\alpha = \frac{1}{2} \tag{3.6}$$

Then : $$G = \frac{1}{2}[d\omega_{ab} + \omega_{af} \wedge \omega^f_{.b} + 4\beta^2\, \omega_a \wedge \omega_b] R^{ab} + \beta\, [d\omega_a + \omega^e_{a.} \wedge \omega_e]\, \gamma^a \tag{3.7a}$$

With the above interpretation of the gauge fields, the first two terms in the first brackets correspond to the usual curvature 2-form $\Omega^a_{.b} = d\omega^a_{.b} + \omega^a_{.c} \wedge \omega^c_{.b}$ , and the second brackets contains the structure equations : $d\omega^a + \omega^a_{.b} \wedge \omega^b = \Sigma^a$ , where $\Sigma^a$ represents the torsion 2-form (B.6) :

$$G = \frac{1}{2}[\Omega_{ab} + 4\beta^2\, \omega_a \wedge \omega_b] R^{ab} + \beta\, \Sigma_a\, \gamma^a \tag{3.7b}$$

The standard minimum gauge field Lagrangian is :

$$L = Tr(G \wedge *G) \tag{3.8}$$

where : $*$ is the Hodge's star operator. Taking into account the $\gamma^a$ matrices properties :

$$Tr(R^{ab}\, R^{cd}) = \frac{N}{4}(\eta^{ad}\, \eta^{bc} - \eta^{ac}\, \eta^{bd}) \quad , \quad Tr(R^{ab}\, \gamma^c) = 0 \quad , \quad Tr(\gamma^a\, \gamma^b) = N\, \eta^{ab} \tag{3.9}$$

where : $N$ is the spinor dimension, one has :

$$L/N = -\beta^2\, \Omega^{ab} \wedge *(\omega_a \wedge \omega_b) - \frac{1}{8}\Omega^{ab} \wedge *\Omega_{ab} \qquad (3.10a)$$
$$-2\beta^4 (\omega^a \wedge \omega^b) \wedge *(\omega_a \wedge \omega_b) + \beta^2\, \eta_{ab}(d\omega^a + \omega^a_{.c} \wedge \omega^c) \wedge *(d\omega^b + \omega^b_{.d} \wedge \omega^d)$$

The first term correspond to the Einstein-Hilbert Lagrangian of General Relativity. The third term represents the contribution of a cosmological constant, since this term is proportional to the volume element. The second term is quadratic and has the form of standard gauge field Lagrangian. The equation (3.10a) can also be re-written :

$$L/N = -\beta^2\, \Omega^{ab} \wedge *(\omega_a \wedge \omega_b) - \frac{1}{8}\Omega^{ab} \wedge *\Omega_{ab} \qquad (3.10b)$$
$$-2\beta^4 (\omega^a \wedge \omega^b) \wedge *(\omega_a \wedge \omega_b) + \beta^2\, \Sigma^a \wedge *\Sigma_a + \mu\,(d\omega^a + \omega^a_{.b} \wedge \omega^b - \Sigma^a)$$

where : $\mu$ is a Lagrange multiplicator.
In (3.10) the torsion is introduced naturally, not as an extra field, this is a direct consequence of the definition (3.2) where $\omega_a$ and $\omega_{ab}$ are independent fields.

Until now, we have discussed the interpretation of the gauge fields introduced in (2.2), but it remains to see how these fields transform. Taking into account the algebra (3.1) , one considers the infinitesimal transformations :

$$S = I + i\varepsilon_a \gamma^a + \varepsilon_{ab} R^{ab}$$

where : $\varepsilon_{ab} = -\varepsilon_{ba}$ et : $|\varepsilon_a|, |\varepsilon_{ab}| \ll 1$ .
The transformation law of gauge filed is : $W' = S^{-1} W S + S^{-1} dS$ which gives : $G' = S^{-1} G S$ and makes the Lagrangian (3.8) invariant. With, at first order : $S^{-1} = I - i\varepsilon_a \gamma^a - \varepsilon_{ab} R^{ab}$
one has, still at first order :

$$W' = W + i\alpha\omega_{ab}\varepsilon_e \left[R^{ab}, \gamma^e\right] + \alpha\, \omega_{ab}\varepsilon_{ef}\left[R^{ab}, R^{ef}\right] + i\beta\omega_a\varepsilon_e\left[\gamma^a, \gamma^e\right]$$
$$+ \beta\,\omega_a \varepsilon_{ef}\left[\gamma^a, R^{ef}\right] + i\, d\varepsilon_e \gamma^e + d\varepsilon_{ef}\, R^{ef}$$

and with the algebra (3.1) :

$$\omega'_a = \omega_a + \varepsilon_{ea}\omega^e - \varepsilon_{ae}\omega^e + \frac{i}{\beta}(d\varepsilon_a + \alpha\,\omega_{a.}^{\ \ e}\varepsilon_e - \alpha\,\omega^e_{.a}\,\varepsilon_e)$$

and with : $\alpha = 1/2$ : $\qquad \omega'_a = \omega_a + \varepsilon_{ea}\omega^e - \varepsilon_{ae}\omega^e + \frac{i}{\beta} D\varepsilon_a$

If : $D\varepsilon_a = 0$ , $\omega^a$ transforms like a vector under an (infinitesimal) rotation whose coefficients are the $\varepsilon_{ef}$ . In that case : $\varepsilon_a = 0$ and the field $\omega_{ab}$ transforms like a gauge field with respect to rotations. The restriction of the gauge transformation makes $\omega^a$ transform like a vector. The 1-forms : $\omega^a = h^a_\alpha\, dx^\alpha$ are directly related to the basis vectors of the local frames.

In the above description $\omega_a$ et $\omega_{ab}$ are independent gauge fields. How does that modifies the Lagrangian of matter fields ? We shall suppose that ordinary matter fields are spinor fields, and therefore we shall consider the covariant derivative of such fields.
Let $\psi$ be a spinor field. The covariant derivative of a spinor field with gauge field is :

$$D\psi = d\psi + \tfrac{1}{4}\omega_{cd}\,\gamma^c\gamma^d\,\psi + \beta\omega_a\gamma^a\,\psi \quad \rightarrow \quad D_\alpha\psi = \partial_\alpha\psi + \tfrac{1}{4}\omega_{cd\alpha}\,\gamma^c\gamma^d\,\psi + \beta h^b_\alpha\,\eta_{bc}\,\gamma^c\,\psi$$

The last term is not present in the usual covariant derivative of a spinor field. The Lagrangian of such a field is : $L = \overline{\psi}\, h^{\alpha}_{a}\, \gamma^{a}\, i\, D_{\alpha}\psi + h.c.$

where : $h.c.$ means : Hermitic conjugate. If one uses the above covariant derivative, the contribution of the unwanted terms is (if $\beta$ is real) :

$$i\,\beta\overline{\psi}\, h^{\alpha}_{a}\,\gamma^{a} h^{b}_{\alpha}\,\eta_{bc}\,\gamma^{c}\,\psi + h.c. = i\,\beta\overline{\psi}\,\gamma^{a}\,\eta_{ac}\,\gamma^{c}\,\psi + h.c. \sim i\,\beta\overline{\psi}\,\psi + h.c. = 0$$

Therefore, the gauge field (3.2) gives the usual spinor field Lagrangian built with the usual covariant derivative : $D_{\alpha}\psi = \partial_{\alpha}\psi + \frac{1}{4}\omega_{cd\alpha}\,\gamma^{c}\gamma^{d}\,\psi$

Remark : the above Lagrangian of a spinor field is built using the generator $\gamma^{a}$ of the algebra (3.1). Why not use the generators of type $R^{ab}$ instead ? A possible Lagrangian could be : $L = \overline{D_{a}\psi}\; R^{ab}\, D_{b}\psi + h.c.$ , which is , taking into account the fact that : $R^{ab} = \frac{1}{2}\gamma^{a}\gamma^{b}$ , $a \neq b$ ,

$$L = \overline{\not{D}\psi}\;.\not{D}\psi - \overline{D_{a}\psi}\;\eta^{ab}\, D_{b}\psi$$

which gives the second order Dirac equation. In conclusion, the gauge field (2.2) does not introduces unwanted terms.

## Summary.

The gauge field (3.2) associated to the algebra (3.1) leads to a gravitational Lagrangian which is more general than the Einstein-Hilbert one, introducing naturally a quadratic term and a cosmological constant. The 1-form fields $\omega^{a}$ and $\omega_{ab}$ are independent of each other, and as a consequence, torsion may exist as an independent field.

# 4. The Dirac operator in $V^{n+m}$ .

## 4.1 The classical Dirac equation with gauge fields : from $V^{n+m}$ to $V^{n}$ .

The evolution of a spinor field in $V^{n+m}$ will be assumed to be governed by the standard « minimum » Lagrangian :

$$L = \overline{\psi}\,\gamma^{a}\, h^{\alpha}_{a}\, i\, D_{\alpha}\psi + (h.c.) + mass\, terms \qquad (4.1)$$

where : $h.c.$ means : hermitic conjugate.

The construction of a possible representation of the Dirac matrices $\gamma^{a}$ is detailed in appendix A . In all what follows $m$ is assumed to be even.

After integration of the Action on $V^{m}$ , one would like to recover the « macroscopic » Dirac equation with gauge fields and gravitational field. In other words, one would like to write :

$$\gamma^{i}\, h^{\mu}_{i}\,(\partial_{\mu} + \Gamma_{jk\mu}\frac{\gamma^{j}\gamma^{k}}{4} - S_{\mu} + W_{\mu})\,\psi = m\,\psi \qquad (4.2)$$

Note that, when we write this equation in this way, we assume that the spinor $\psi$ is a « multiplet » of : $q_{m} = 2^{\frac{m}{2}}$ spinors $\psi_{i}$ of $V^{n}$ of the form :

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{q_m} \end{pmatrix} \qquad (4.3)$$

In the Dirac equation (4.2), the gauge fields acts on the multiplets and not on the components of each $\psi_i$ . In other words, in equation (4.2) , the $\gamma^i$ are supposed to be of the form : $I_{q_m} \otimes \gamma^i_n$ , where we have written : $\gamma^a_p$ to mean it is the Dirac matrice number $a$ defined in a space-time of dimension $p$ .

The appendix A shows that is is possible to « diagonalise » the Dirac matrices of type $\gamma^i_{n+m}$ , $0 \le i < n$ , in the form : $I_{q_m} \otimes \gamma^i_n$ (relations (A.23) , (A.24) , (A.27) , (A.17) ) .

The commutation relations : $\left[\gamma^i , \gamma^r \gamma^s\right] = 0$ et : $\left[\gamma^i \gamma^j , \gamma^r \gamma^s\right] = 0$ show that the : $\dfrac{\gamma^r \gamma^s}{2}$ are good candidates to represent the $SO(m)$ Lie algebra generators. At last, note that, from now on, we shall suppose that the conditions (2.10) are satisfied.

## 4.2 Decomposition of the Dirac operator.

From (1.15) , one has : $\Gamma_{abc} = \overline{\Gamma}_{abc} + S_{abc} + S_{bca} + S_{cba}$

Using (2.18), the Dirac operator (2.15) is written :

$$\begin{aligned} \not{D} = \gamma^a h^\alpha_a D_\alpha \psi = \gamma^i h^\mu_i (\partial_\mu + \overline{\Gamma}_{jk\mu} \frac{\gamma^j \gamma^k}{4} + \overline{\Gamma}_{jr\mu} \frac{\gamma^j \gamma^r}{2} + \overline{\Gamma}_{rs\mu} \frac{\gamma^r \gamma^s}{4}) \psi \\ + \quad \gamma^i h^\varphi_i (\partial_\varphi + \overline{\Gamma}_{jk\varphi} \frac{\gamma^j \gamma^k}{4} + \overline{\Gamma}_{jr\varphi} \frac{\gamma^j \gamma^r}{2} + \overline{\Gamma}_{rs\varphi} \frac{\gamma^r \gamma^s}{4}) \psi \qquad (4.4) \\ + \quad \gamma^r h^\varphi_r (\partial_\varphi + \overline{\Gamma}_{st\varphi} \frac{\gamma^s \gamma^t}{4} + \overline{\Gamma}_{jk\varphi} \frac{\gamma^j \gamma^k}{4} + \overline{\Gamma}_{js\varphi} \frac{\gamma^j \gamma^s}{2}) \psi + S_{abc} \frac{\gamma^a \gamma^b \gamma^c}{4} \end{aligned}$$

where, in the last term : $a \ne b \ne c \ne a$ (only the totally anti-symmetric part of the torsion tensor is taken into account).

Grouping the terms 3 et 7 , (4.4) is :

$$\begin{aligned} \not{D} = \gamma^i h^\mu_i (\partial_\mu + \overline{\Gamma}_{jk\mu} \frac{\gamma^j \gamma^k}{4} + \overline{\Gamma}_{rs\mu} \frac{\gamma^r \gamma^s}{4}) \psi \; + D_m \psi \\ + \overline{\Gamma}_{jri} \frac{\gamma^i \gamma^j \gamma^r}{2} \; \psi + \gamma^i h^\varphi_i (\partial_\varphi + \overline{\Gamma}_{jk\varphi} \frac{\gamma^j \gamma^k}{4} + \overline{\Gamma}_{rs\varphi} \frac{\gamma^r \gamma^s}{4}) \psi \qquad (4.5) \\ + ( \overline{\Gamma}_{jkr} \frac{\gamma^r \gamma^j \gamma^k}{4} + \overline{\Gamma}_{jsr} \frac{\gamma^r \gamma^j \gamma^s}{2} ) \psi + S_{abc} \frac{\gamma^a \gamma^b \gamma^c}{4} \end{aligned}$$

where we have defined : $$D_m = \gamma^r h^\varphi_r (\partial_\varphi + \overline{\Gamma}_{st\varphi} \frac{\gamma^s \gamma^t}{4}) \qquad (4.6)$$

$D_m$ is Dirac operator on $V^m$ without torsion, but it is not the Dirac equation on $V^m$ because it does not include the time coordinate . This operator will be studied in chapter 6 when the mass term will be discussed .

The first term of the second line and the first term of the third line give, using (1.16f) :

$$\overline{\Gamma}_{jri}\frac{\gamma^i\gamma^j\gamma^r}{2}+\overline{\Gamma}_{jkr}\frac{\gamma^r\gamma^j\gamma^k}{4}=\overline{\Gamma}_{jri}\frac{\gamma^i\gamma^j\gamma^r}{2}+\overline{\Gamma}_{rkj}\frac{\gamma^r\gamma^j\gamma^k}{4}$$
$$=\left(\overline{\Gamma}_{krj}+\frac{1}{2}\overline{\Gamma}_{rkj}\right)\frac{\gamma^r\gamma^j\gamma^k}{2}+\Gamma^k_{.rk}\frac{\gamma^r}{2}$$

where : $j\neq k$ in the first term on the right . Finally , using (1.16f) again :

$$\overline{\Gamma}_{jri}\frac{\gamma^i\gamma^j\gamma^r}{2}+\overline{\Gamma}_{jkr}\frac{\gamma^r\gamma^j\gamma^k}{4}=-\overline{\Gamma}_{jkr}\frac{\gamma^r\gamma^j\gamma^k}{4}+\overline{\Gamma}^k_{.rk}\frac{\gamma^r}{2}$$

for orthonormal frames, one has with (1.16f) : $\overline{\Gamma}_{krk}=-\overline{\Gamma}_{rkk}=-\overline{\Gamma}_{kkr}=0$ . Then :

$$\overline{\Gamma}_{jri}\frac{\gamma^i\gamma^j\gamma^r}{2}+\overline{\Gamma}_{jkr}\frac{\gamma^r\gamma^j\gamma^k}{4}=-\overline{\Gamma}_{jkr}\frac{\gamma^r\gamma^j\gamma^k}{4}\tag{4.7}$$

The term before the last in (4.5) is :

$$\overline{\Gamma}_{jsr}\frac{\gamma^r\gamma^j\gamma^s}{2}=-\overline{\Gamma}_{sjr}\frac{\gamma^r\gamma^j\gamma^s}{2}=-\frac{1}{2}\left(\overline{\Gamma}_{sjr}\frac{\gamma^r\gamma^j\gamma^s}{2}+\overline{\Gamma}_{rjs}\frac{\gamma^s\gamma^j\gamma^r}{2}\right)$$

then , by permutation of the Dirac matrices in the second term, and with (1.16f) :

$$\overline{\Gamma}_{jsr}\frac{\gamma^r\gamma^j\gamma^s}{2}=\overline{\Gamma}^r_{.jr}\frac{\gamma^j}{2}\tag{4.8}$$

The equation (4.5) becomes :

$$\not{D}=\gamma^i h^\mu_i(\partial_\mu+\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}+\overline{\Gamma}_{rs\mu}\frac{\gamma^r\gamma^s}{4})\psi+D_m\psi\tag{4.9}$$
$$+\gamma^i h^\varphi_i(\partial_\varphi+\overline{\Gamma}_{jk\varphi}\frac{\gamma^j\gamma^k}{4}+\overline{\Gamma}_{rs\varphi}\frac{\gamma^r\gamma^s}{4})\psi-(\overline{\Gamma}_{jkr}\frac{\gamma^r\gamma^j\gamma^k}{4}-\overline{\Gamma}^r_{.jr}\frac{\gamma^j}{2})\psi$$
$$+S_{abc}\frac{\gamma^a\gamma^b\gamma^c}{4}$$

where : $\overline{\Gamma}_{jkr}=h^\varphi_r\,\overline{\Gamma}_{jk\varphi}$ .

In order to assess the relative importance of the terms in (4.9), we shall write them as functions of the fields : $h^r_\mu$ .

From (1.16k), and with condition (1.19a), we have : $2\,\overline{\Gamma}_{ijr}=\eta_{rs}\,C^s_{.ij}$

With (1.7) , (1.18d) is :

$$C^s_{.ij}=h^\nu_i h^\rho_j\left\{\overline{F}^s_{\rho\nu}+h^t_\nu h^\varphi_t\,\overline{F}^s_{\varphi\rho}-h^t_\rho h^\varphi_t\,\overline{F}^s_{\varphi\nu}+h^r_\nu h^\omega_r h^t_\rho h^\varphi_t)\overline{F}^s_{\varphi\omega}\right\}$$

we have also :
$$C^t_{.rs}=h^\varphi_r h^\tau_s\,\overline{F}^t_{\tau\varphi}\tag{4.10}$$

therefore :
$$C^s_{.ij}=h^\nu_i h^\rho_j\left\{\overline{F}^s_{\rho\nu}+h^r_\nu h^t_\rho C^s_{.rt}+h^t_\nu h^\varphi_t\,\overline{F}^s_{\varphi\rho}-h^t_\rho h^\varphi_t\,\overline{F}^s_{\varphi\nu}\right\}\tag{4.11}$$

In the same way (1.18c) is :

$$C^r_{.iu}=h^\nu_i h^\varphi_u\,\overline{F}^r_{\varphi\nu}-h^\nu_i h^t_\nu h^\varphi_t h^\tau_u\,\overline{F}^r_{\tau\varphi}=h^\nu_i(h^\varphi_u\,\overline{F}^r_{\varphi\nu}-h^t_\nu C^r_{.tu})\tag{4.12}$$

and (1.16i) :

$$2\overline{\Gamma}_{rsi}=\eta_{rt}C^t_{.is}-\eta_{st}C^t_{.ir}=h^\nu_i\left((\eta_{rt}h^\varphi_s-\eta_{st}h^\varphi_r)\,\overline{F}^t_{\varphi\nu}-h^u_\nu(C_{rus}-C_{sur})\right)\tag{4.13}$$

$$2\overline{\Gamma}_{irs} = \eta_{rt}\, C^{t}_{.is} + \eta_{st}\, C^{t}_{.ir} = h^{\nu}_{i}\left( (\eta_{rt}\, h^{\varphi}_{s} + \eta_{st}\, h^{\varphi}_{r})\, \overline{F}^{t}_{\varphi\nu} - h^{u}_{\nu}\,(C_{rus} + C_{sur}) \right) \qquad (4.14)$$

With (1.15g) one obtains :

$$2\overline{\Gamma}_{rs\mu} = 2\overline{\Gamma}_{rsi}\, h^{i}_{\mu} + 2\overline{\Gamma}_{rst}\, h^{t}_{\mu} = (\eta_{rt}\, h^{\varphi}_{s} - \eta_{st}\, h^{\varphi}_{r})\, \overline{F}^{t}_{\varphi\mu} + h^{t}_{\mu}\, C_{trs} \qquad (4.15)$$

The basic hypothesis was that the space-time manifold was locally of the form $V = V^{n} \otimes V^{m}$ where : $V^{m}$ is a compact space invariant by the action of a group of motion $G$ . We shall set :

$$h^{r}_{\varphi} = q(x^{\mu})\, \overline{h}^{r}_{\varphi}(x^{\tau}) \qquad (4.16)$$

where the fields : $\overline{h}^{r}_{\varphi}$ are the components of the differential forms associated to the orthonormal local frames for a manifold $V^{m}$ of unit curvature radius , and where : $q(x^{\mu})$ represents the curvature radius of $V^{m}$ at the point $\{x^{\mu}\}$ of $V^{n}$ . With (1.7) one then has :

$$h^{r}_{\varphi}\, h^{\varphi}_{s} = q(x^{\mu})\, \overline{h}^{r}_{\varphi}\, h^{\varphi}_{s} = \delta^{r}_{s} \quad \rightarrow \quad h^{\varphi}_{s} = \frac{1}{q}\, \overline{h}^{\varphi}_{s}(x^{\tau}) \qquad (4.17a)$$

or , like in (1.7) :
$$\overline{h}^{r}_{\varphi}\, \overline{h}^{\varphi}_{s} = \delta^{r}_{s} \quad , \quad \overline{h}^{r}_{\varphi}\, \overline{h}^{\tau}_{r} = \delta^{\tau}_{\varphi} \qquad (4.17b)$$

We now gather a few results useful for the coming calculations :

$$C^{s}_{.tu} = \frac{1}{q}\,(\overline{h}^{\varphi}_{t}\, \partial_{\varphi} \overline{h}^{\tau}_{u} - \overline{h}^{\varphi}_{u}\, \partial_{\varphi} \overline{h}^{\tau}_{t})\, h^{s}_{\tau} \equiv \frac{1}{q}\, \overline{C}^{s}_{.tu} \qquad (4.18)$$

$$2\overline{\Gamma}_{rs\mu} = \frac{1}{q}\left( (\eta_{rt}\, \overline{h}^{\varphi}_{s} - \eta_{st}\, \overline{h}^{\varphi}_{r})\, \partial_{\varphi} h^{t}_{\mu} + h^{t}_{\mu}\, \overline{C}_{trs} \right) \qquad (4.19)$$

$$h^{\varphi}_{i}\, \overline{\Gamma}_{jk\varphi} = h^{\varphi}_{i}\, h^{r}_{\varphi}\, \overline{\Gamma}_{jkr} = -(h^{\mu}_{i}\, h^{t}_{\mu}\, h^{\varphi}_{t})\, h^{r}_{\varphi}\, \overline{\Gamma}_{jkr} = -h^{\mu}_{i}\, h^{r}_{\mu}\, \overline{\Gamma}_{jkr} \qquad (4.20)$$

this term, which is the ninth of (4.9) contributes, as well as the second of (4.9) to terms of the form : $\gamma^{i}\,\gamma^{j}\,\gamma^{k}$ .

$$2\,\overline{\Gamma}_{jkr} = \eta_{rs}\, h^{\nu}_{j}\, h^{\rho}_{k} \left\{ \overline{F}^{s}_{\rho\nu} + h^{t}_{\nu}\, h^{u}_{\rho}\, C^{s}_{.tu} + \frac{1}{q}\, \overline{h}^{\varphi}_{t}\,(h^{t}_{\nu}\, \partial_{\varphi} h^{s}_{\rho} - h^{t}_{\rho}\, \partial_{\varphi} h^{s}_{\nu}) + \frac{1}{q}\,(q_{\nu}\, h^{s}_{\rho} - q_{\rho}\, h^{s}_{\nu}) \right\} \qquad (4.21)$$

With (4.16) :
$$h^{r}_{\nu} \sim O(q) \qquad (4.22)$$

and :
$$\overline{\Gamma}_{rs\mu} \sim O(1) \qquad (4.23a)$$

$$\overline{\Gamma}_{jkr} \sim O(q) \qquad (4.23b)$$

$$h^{\varphi}_{i}\, \overline{\Gamma}_{jk\varphi} \sim O(q^{2}) \qquad (4.23c)$$

$$h^{\varphi}_{k} = -h^{\mu}_{k}\, h^{r}_{\mu}\, h^{\varphi}_{r} \;\rightarrow\; h^{\varphi}_{k} \sim O(1) \qquad (4.23d)$$

from (4.12) and (4.18) :
$$C_{ris} \sim O(1) \qquad (4.23e)$$

from (4.14) and (4.18) :
$$\overline{\Gamma}_{irs} \sim O(1) \qquad (4.23f)$$

The seventh term of (4.9) is $\sim h^{\varphi}_{i}\, \overline{\Gamma}_{rs\varphi}$ , with : $\overline{\Gamma}_{rs\varphi} = h^{a}_{\varphi}\, \overline{\Gamma}_{rsa} = h^{t}_{\varphi}\, \overline{\Gamma}_{rst} = q\, \overline{h}^{t}_{\varphi}\, \overline{\Gamma}_{rst}$

then :
$$h^{\varphi}_{i}\, \overline{\Gamma}_{rs\varphi} = -h^{\mu}_{i}\, h^{t}_{\mu}\, h^{\varphi}_{t}\, q\, \overline{h}^{u}_{\varphi}\, \overline{\Gamma}_{rsu} = -h^{\mu}_{i}\, h^{t}_{\mu}\, \overline{\Gamma}_{rst}$$

with : $$2\,\overline{\Gamma}_{rst} = C_{trs} - C_{str} - C_{rst} = \frac{1}{q}\left(\overline{C}_{trs} - \overline{C}_{str} - \overline{C}_{rst}\right) \tag{4.24}$$

therefore: $h_i^{\varphi}\,\overline{\Gamma}_{rs\varphi} \sim O(1)$ , and this term can be grouped with the third of (4.9) to give : $\overline{\Gamma}_{rsi}\,\dfrac{\gamma^i\gamma^r\gamma^s}{4}$ . With (4.13) :

$$2\overline{\Gamma}_{rsi}\,\gamma^i\gamma^r\gamma^s = \gamma^i h_i^{\nu}\left((\eta_{rt}\,h_s^{\varphi} - \eta_{st}\,h_r^{\varphi})\,\overline{F}^{t}_{\varphi\nu} - h_\nu^u\,(C_{rus} - C_{sur})\right)\gamma^r\gamma^s \tag{4.25}$$

The « macroscopic" Dirac operator (4.9) is therefore :

$$\not{D} = \gamma^i\,h_i^{\mu}\left(\partial_\mu + \overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4} + \left(\frac{1}{2}(\eta_{rt}\,h_s^{\varphi} - \eta_{st}\,h_r^{\varphi})\,\overline{F}^{t}_{\varphi\mu} - h_\mu^t\,C_{rts}\right)\frac{\gamma^r\gamma^s}{4}\right)\psi$$
$$+\gamma^i\,h_i^{\varphi}\,\partial_\varphi\psi + \overline{\Gamma}^{r}_{.jr}\frac{\gamma^j}{2}\psi + D_m\psi + S_{abc}\frac{\gamma^a\gamma^b\gamma^c}{4} \tag{4.26}$$

where : $r \neq s$ in the first line, and $a \neq b \neq c \neq a$ in the last term.

Taking into account (1.7) , one can put together the last term of the first line and the first of the second line : $\gamma^i\,h_i^{\varphi}\,\partial_\varphi - \gamma^i\,h_i^{\mu}\,h_\mu^t\,C_{rts}\dfrac{\gamma^r\gamma^s}{4} = -\gamma^i\,h_i^{\mu}\,h_\mu^t\,(h_t^{\varphi}\,\partial_\varphi + C_{rts}\dfrac{\gamma^r\gamma^s}{4})$

We define, using , (4.18) : $$\widehat{T}_t = \overline{h}_t^{\varphi}\,\partial_\varphi + \overline{C}_{rts}\frac{\gamma^r\gamma^s}{4} \tag{4.27}$$

The operator $\widehat{T}_t$ has the same properties as $\widehat{T}_x$ , and satisfies the relations (2.8) , if the $\overline{h}_t^{\varphi}\,\partial_\varphi$ are associated to rigid motions. With (1.11) this condition is :

$$C_{rts} + C_{str} = 0 \tag{4.28}$$

In order to consider the operators $\widehat{T}_t$ as the operators of the Lie algebra of a group $Q$ , $V^m$ must be the equivalent of the parameter space of this group. $G$ is an invariance group of $V^m$ , it is sufficient that $Q$ is the quotient group of $G$ by the (invariant) stability group of $V^m$ .

Note : if the Lie algebra of $G$ is semi simple and compact, the Killing form of the real Lie algebra is negative definite by definition, and the structure constants are totally anti-symmetric. In the following we shall assume that $G$ is a simple group.

From (4.22) we define : $$h_\nu^r = -q(x^\mu)\,A_\nu^r(x^\rho) \tag{4.29a}$$

then , using (4.23d) : $$h_k^{\varphi} = h_k^{\mu}\,A_\mu^r\,\overline{h}_r^{\varphi} = A_k^r\,\overline{h}_r^{\varphi} \tag{4.29b}$$

This hypothesis is more constraining than (4.22), it means that $h_\nu^r$ does not depend on the internal coordinates of $V^m$ .

Then : $$\gamma^i\,h_i^{\varphi}\,\partial_\varphi - \gamma^i\,h_i^{\mu}\,h_\mu^t\,C_{rts}\frac{\gamma^r\gamma^s}{4} = \gamma^i\,h_i^{\mu}\,A_\mu^t\,\widehat{T}_t$$

represents the field $A_\nu^r(x^\rho)$ contribution. This field transforms like $W_\mu^x$ in section 1.1.

Now we consider the second term of (4.26) :

$$\overline{\Gamma}_{jk\mu} = h_\mu^i\,\overline{\Gamma}_{jki} + h_\mu^r\,\overline{\Gamma}_{jkr} = h_\mu^i\,\overline{\Gamma}_{jki} + O(q^2)$$

from (1.16h) and (1.19a) , $\overline{\Gamma}_{jki}$ and $h_i^{\mu}$ do not depend on the $\{x^{\nu}\}$ , then, at the macroscopic level :

$$\left[\gamma^i h_i^{\mu}(\partial_{\mu}+\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}) \ , \ \widehat{T}_t\right]=0 \tag{4.30}$$

With the hypothesis (4.29), equation (4.26) becomes :

$$\not{D}=\gamma^i h_i^{\mu}\left(\partial_{\mu}+\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}+A_{\mu}^t\widehat{T}_t\right)\psi+\overline{\Gamma}^r_{.jr}\frac{\gamma^j}{2}\psi+D_m\psi+S_{abc}\frac{\gamma^a\gamma^b\gamma^c}{4}$$

Now, from (1.15f) and (4.28) : $\overline{\Gamma}^r_{.jr}=-C^r_{.jr}=0$ , and it remains, at the « macroscopic » level :

$$\not{D}=\gamma^i h_i^{\mu}\left(\partial_{\mu}+\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}+A_{\mu}^t\widehat{T}_t\right)\psi+D_m\psi+S_{abc}\frac{\gamma^a\gamma^b\gamma^c}{4} \tag{4.31}$$

where : $a\neq b\neq c\neq a$ in the last term.
In order to obtain the Dirac equation (4.2) from (4.31) , one must « diagonalise » the first term of (4.31). This can be done by multiplying (4.31) by the operator (A24).

## Summary.

In order to put the Dirac operator $\not{D}$ in the form (4.2) we have done 2 hypothesis : the hypothesis (4.29) : $h_{\nu}^r=-q(x^{\mu})A_{\nu}^r(x^{\rho})$ , and that $V^m$ is equivalent to the parameter space of $G$ , with the contraints (4.28) . With this condition on $V^m$ , the operators $\widehat{T}_t$ can be considered as belonging to a representation of the Lie algebra of $G$ , moreover they commute with the « spatial » part of the Dirac operator (4.31).

# 5 . Gauge field Lagrangian.

## 5.1 Curvature tensor decomposition.

The curvature 2-form is :

$$\Omega_{ab}=d(\Gamma_{abg}\omega^g)+\Gamma_{aef}\omega^f\wedge\Gamma^e_{.bg}\omega^g$$
$$=d\Gamma_{abg}\wedge\omega^g+\Gamma_{abg}(\Sigma^g_{.cd}-\Gamma^g_{.dc})\,\omega^c\wedge\omega^d+\Gamma_{aef}\Gamma^e_{.bg}\omega^f\wedge\omega^g$$

symmetrizing and using (1.15a) :

$$2\,\Omega_{ab}=\left(h_f^{\alpha}\partial_{\alpha}\Gamma_{abg}-h_g^{\alpha}\partial_{\alpha}\Gamma_{abf}+\Gamma_{aef}\Gamma^e_{.bg}-\Gamma_{aeg}\Gamma^e_{.bf}-\Gamma_{abe}C^e_{.fg}\right)\omega^f\wedge\omega^g \tag{5.1}$$

which is valid with or without torsion. With the definitions (1.15) the connexion is re-written :

$$\Gamma_{abc}=\overline{\Gamma}_{abc}+\overline{S}_{abc} \tag{5.2}$$

where : $\overline{S}_{abc}$ is the contorsion tensor and : $\overline{\Gamma}_{abc}$ is the part of the connexion depending on the Christoffel symbols. The curvature tensor is then :

$$2\,\Omega_{ab}=\left(h_f^{\alpha}\partial_{\alpha}\overline{\Gamma}_{abg}-h_g^{\alpha}\partial_{\alpha}\overline{\Gamma}_{abf}+\overline{\Gamma}_{aef}\overline{\Gamma}^e_{.bg}-\overline{\Gamma}_{aeg}\overline{\Gamma}^e_{.bf}-\overline{\Gamma}_{abe}C^e_{.fg}\right)\omega^f\wedge\omega^g$$
$$+\left(h_f^{\alpha}\partial_{\alpha}\overline{S}_{abg}-h_g^{\alpha}\partial_{\alpha}\overline{S}_{abf}+\overline{\Gamma}_{aef}\overline{S}^e_{.bg}+\overline{S}_{aef}\overline{\Gamma}^e_{.bg}-\overline{\Gamma}_{aeg}\overline{S}^e_{.bf}-\overline{S}_{aeg}\overline{\Gamma}^e_{.bf}\right)\omega^f\wedge\omega^g$$
$$-\overline{S}_{abe}C^e_{.fg}+\left(\overline{S}_{aef}\overline{S}^e_{.bg}-\overline{S}_{aeg}\overline{S}^e_{.bf}\right)\omega^f\wedge\omega^g$$

we set :

$$\overline{D}_f \overline{S}_{abg} = h_f^\alpha \partial_\alpha \overline{S}_{abg} - \overline{\Gamma}^e_{.af} \overline{S}_{ebg} - \overline{\Gamma}^e_{.bf} \overline{S}_{aeg} - \overline{\Gamma}^e_{.gf} \overline{S}_{abe} \tag{5.3}$$

then , with (1.15h) :

$$2\,\Omega_{ab} = \left( h_f^\alpha \partial_\alpha \overline{\Gamma}_{abg} - h_g^\alpha \partial_\alpha \overline{\Gamma}_{abf} + \overline{\Gamma}_{aef} \overline{\Gamma}^e_{.bg} - \overline{\Gamma}_{aeg} \overline{\Gamma}^e_{.bf} - \overline{\Gamma}_{abe} C^e_{.fg} \right) \omega^f \wedge \omega^g$$
$$+ \left( \overline{D}_f \overline{S}_{abg} - \overline{D}_g \overline{S}_{abf} + \overline{S}_{aef} \overline{S}^e_{.bg} - \overline{S}_{aeg} \overline{S}^e_{.bf} \right) \omega^f \wedge \omega^g \tag{5.4}$$

which separates the torsion terms.

In the following we shall calculate the part of the curvature tensor corresponding to the first line of (5.4), or equivalently, we shall suppose for a while, that the the torsion is null. We now write separately the curvature tensor components.

$$\begin{aligned} 2\,\Omega_{ij} = &\left( h_k^\alpha \partial_\alpha \Gamma_{ijl} + \Gamma_{iek} \Gamma^e_{.jl} - (k \leftrightarrow l) - \Gamma_{ije} C^e_{.kl} \right) \omega^k \wedge \omega^l \\ &+ 2 \left( h_k^\alpha \partial_\alpha \Gamma_{ijr} + \Gamma_{iek} \Gamma^e_{.jr} - (k \leftrightarrow r) - \Gamma_{ije} C^e_{.kr} \right) \omega^k \wedge \omega^r \\ &+ \left( h_r^\alpha \partial_\alpha \Gamma_{ijs} + \Gamma_{ier} \Gamma^e_{.js} - (r \leftrightarrow s) - \Gamma_{ije} C^e_{.rs} \right) \omega^r \wedge \omega^s \end{aligned} \tag{5.5}$$

$$\begin{aligned} 2\,\Omega_{it} = &\left( h_k^\alpha \partial_\alpha \Gamma_{itl} + \Gamma_{iek} \Gamma^e_{.tl} - (k \leftrightarrow l) - \Gamma_{ite} C^e_{.kl} \right) \omega^k \wedge \omega^l \\ &+ 2 \left( h_k^\alpha \partial_\alpha \Gamma_{itr} + \Gamma_{iek} \Gamma^e_{.tr} - (k \leftrightarrow r) - \Gamma_{ite} C^e_{.kr} \right) \omega^k \wedge \omega^r \\ &+ \left( h_r^\alpha \partial_\alpha \Gamma_{its} + \Gamma_{ier} \Gamma^e_{.ts} - (r \leftrightarrow s) - \Gamma_{ite} C^e_{.rs} \right) \omega^r \wedge \omega^s \end{aligned} \tag{5.6}$$

$$\begin{aligned} 2\,\Omega_{tu} = &\left( h_k^\alpha \partial_\alpha \Gamma_{tul} + \Gamma_{tek} \Gamma^e_{.ul} - (k \leftrightarrow l) - \Gamma_{tue} C^e_{.kl} \right) \omega^k \wedge \omega^l \\ &+ 2 \left( h_k^\alpha \partial_\alpha \Gamma_{tur} + \Gamma_{tek} \Gamma^e_{.ur} - (k \leftrightarrow r) - \Gamma_{tue} C^e_{.kr} \right) \omega^k \wedge \omega^r \\ &+ \left( h_r^\alpha \partial_\alpha \Gamma_{tus} + \Gamma_{ter} \Gamma^e_{.us} - (r \leftrightarrow s) - \Gamma_{tue} C^e_{.rs} \right) \omega^r \wedge \omega^s \end{aligned} \tag{5.7}$$

## 5.2 Computation of the components.

In order to compute the components of the curvature tensor we shall need the following elements :

$$C^r_{.kl} = [h_k, h_l]^\beta h_\beta^r = [h_k, h_l]^\nu h_\nu^r + [h_k, h_l]^\varphi h_\varphi^r$$

$$C^r_{.kl} = (h_k^\mu \partial_\mu h_l^\nu - h_l^\mu \partial_\mu h_k^\nu) h_\nu^r + (h_k^\mu \partial_\mu h_l^\varphi - h_l^\mu \partial_\mu h_k^\varphi) h_\varphi^r + (h_k^\tau \partial_\tau h_l^\varphi - h_l^\tau \partial_\tau h_k^\varphi) h_\varphi^r$$

with : $C^j_{.kl} = [h_k, h_l]^\beta h_\beta^j = [h_k, h_l]^\nu h_\nu^j$ , and (4.29) , one has :

$$C^r_{.kl} = -q\, C^j_{.kl} h_j^\nu A_\nu^r + q (h_k^\mu \partial_\mu A_l^r - h_l^\mu \partial_\mu A_k^r) + q (A_l^t h_k^\tau \partial_\tau \overline{h}_t^\varphi - A_k^t h_l^\tau \partial_\tau \overline{h}_t^\varphi) \overline{h}_\varphi^r$$

$$C^r_{.kl} = -q\, C^j_{.kl} h_j^\nu A_\nu^r + q \left( (h_k^\mu \partial_\mu A_l^r - h_l^\mu \partial_\mu A_k^r) + A_l^t A_k^s (\overline{h}_s^\tau \partial_\tau \overline{h}_t^\varphi - \overline{h}_t^\tau \partial_\tau \overline{h}_s^\varphi) \overline{h}_\varphi^r \right)$$

$$C^r_{.kl} = -q\, C^j_{.kl} h_j^\nu A_\nu^r + q \left( h_k^\mu \partial_\mu (h_l^\nu A_\nu^r) - h_l^\mu \partial_\mu (h_k^\nu A_\nu^r) + A_l^t A_k^s \overline{C}^r_{.st} \right)$$

and finally :
$$C^r_{.kl} = q\, G^r_{lk} = q\, h_k^\mu h_l^\nu G^r_{\mu\nu} \tag{5.8}$$

where :
$$G^t_{\mu\nu} = \partial_\mu A_\nu^t - \partial_\nu A_\mu^t + A_\mu^r A_\nu^s \overline{C}^t_{.rs} \tag{5.9}$$

With (1.16k) :
$$2\,\overline{\Gamma}_{ijr} = q\, \eta_{rs}\, h_i^\mu h_j^\nu G^s_{\mu\nu} = q\, G_{rij} \tag{5.10a}$$

which satisfies :
$$\partial_\varphi \overline{\Gamma}_{jkr} = 0 \tag{5.10b}$$

To obtain $\overline{\Gamma}_{irs}$ we use (4.13) . In this expression the last term is null because of condition (4.28), then : $2\overline{\Gamma}_{irs} = -\, h_i^{\nu} \dfrac{1}{q} (\eta_{rt}\, \overline{h}_s^{\varphi} + \eta_{st}\, \overline{h}_r^{\varphi})\, \partial_\nu (q\, \overline{h}_\varphi^{t})$

$$\overline{\Gamma}_{irs} = -\, h_i^{\nu} \frac{q_\nu}{q}\, \eta_{rs} \tag{5.11}$$

with (4.28) , (4.24) becomes : $$2\, \overline{\Gamma}_{rst} = -\frac{1}{q}\, \overline{C}_{rst} \tag{5.12}$$

(4.13) becomes : $$\overline{\Gamma}_{rsi} = h_i^{\mu}\, A_\mu^t\, \overline{C}_{rts} \tag{5.13}$$

$$\begin{aligned} C^{s}_{.kr} &= \left[h_k , h_r\right]^{\beta} h_\beta^s = (h_k^\alpha\, \partial_\alpha h_r^\beta - h_r^\alpha\, \partial_\alpha h_k^\beta)\, h_\beta^s \\ &= (h_k^\mu\, \partial_\mu h_r^\tau + h_k^\varphi\, \partial_\varphi h_r^\tau - h_r^\varphi\, \partial_\varphi h_k^\tau)\, h_\tau^s - h_r^\varphi\, \partial_\varphi h_k^\nu\, h_\nu^s \end{aligned}$$

and with (4.23d) and (4.29) : $h_k^\tau = -h_k^\mu\, h_\mu^u\, h_u^\tau = h_k^\mu\, A_\mu^u\, \overline{h}_u^{\tau}$ , then :

$$C^{s}_{.kr} = -\frac{h_k^\mu\, q_\mu}{q}\, \delta_r^s + h_k^\mu\, A_\mu^u\, \overline{C}^{s}_{.ur} \tag{5.14}$$

We can now come back to the curvature tensor. Using the relations (1.19) , (5.8) et (5.10) , and setting : $2\, \Omega_{n,ij} = \left(h_k^\alpha\, \partial_\alpha \Gamma_{ijl} + \Gamma_{imk}\, \Gamma^{m}_{.jl} - (k \leftrightarrow l) - \Gamma_{ijm}\, C^{m}_{.kl}\right) \omega^k \wedge \omega^l$
which represents the curvature on $V^n$ only , one obtains :

$$\begin{aligned} 2\, \Omega_{ij} = 2\, \Omega_{n,ij} &- \frac{q^2}{2} \left( G^{r}_{ki}\, G_{rlj} + G^{r}_{ij}\, G_{rkl}\right) \omega^k \wedge \omega^l \\ &+ q \left(\partial_k G_{rij} + \Gamma_{imk}\, G^{\;m}_{r.j} + \Gamma_{jmk}\, G^{\;m}_{ri.} - A_k^t\, \overline{C}^{s}_{.tr}\, G_{sij}\right) \omega^k \wedge \omega^r \\ &- (2\, q_k\, G_{rij} + q_i\, G_{rkj} - q_j\, G_{rki})\;\; \omega^k \wedge \omega^r \\ &+ \frac{q^2}{2} G_{rim}\, G^{\;m}_{s.j}\;\; \omega^r \wedge \omega^s - \frac{1}{2}\; G^{t}_{ij}\, \overline{C}_{trs}\;\; \omega^r \wedge \omega^s \end{aligned} \tag{5.15}$$

The second line of this equation is a covariant derivative .

In the same way, using the relation (1.15h) for the first line of (5.6) and the relation (4.28) for the second and third lines , one obtains :

$$\begin{aligned} 2\Omega_{it} = &\left(\frac{q}{2}\left(D_k G_{til} - D_l G_{tik}\right) + q_i\, G_{tkl}\right) \omega^k \wedge \omega^l \\ &- 2\left(\frac{\eta_{rt}}{q} D_k q_i + \frac{1}{4} G^{s}_{.ik}\, \overline{C}_{str}\right) \omega^k \wedge \omega^r + \frac{q^2}{2}\left(G_{rim}\, G^{\;\;m}_{tk.}\right) \omega^k \wedge \omega^r \\ &+ \frac{q^m}{2}\left(G_{sim}\, \eta_{tr} - G_{rim}\, \eta_{ts}\right) \omega^r \wedge \omega^s \end{aligned} \tag{5.16}$$

where : $D_k q_i = h_k^\mu\, \partial_\mu q_i - \Gamma^{m}_{.ik}\, q_m$ , and : $D_k G_{rij} = \partial_k G_{rij} + \Gamma_{imk}\, G^{\;m}_{r.j} + \Gamma_{jmk}\, G^{\;m}_{ri.} - A_k^t\, \overline{C}^{s}_{.tr}\, G_{sij}$
like in the second line of (5.15) .

In order to calculate (5.7) we use the Jacobi's identities , then :

$$2\Omega_{tu} = \left( \frac{1}{2} G^{z}_{kl} \overline{C}_{tzu} + \frac{q^2}{4} (G_{tlm} G_{uk.}^{\;\;\;m} - G_{tkm} G_{ul.}^{\;\;\;m}) \right) \omega^k \wedge \omega^l$$
$$+ q^m (G_{tmk} \eta_{ur} - G_{umk} \eta_{tr}) \omega^k \wedge \omega^r \qquad (5.17)$$
$$+ \frac{q_m q^m}{q^2} (\eta_{ru} \eta_{ts} - \eta_{rt} \eta_{us}) \omega^r \wedge \omega^s + 2\, \overline{\Omega}_{m,turs}\, \omega^r \wedge \omega^s$$

where : $2\overline{\Omega}_{m,turs} = h_r^{\varphi} \partial_{\varphi} \overline{\Gamma}_{tus} + \overline{\Gamma}_{tqr} \overline{\Gamma}^{q}_{.us} - (r \leftrightarrow s) - \overline{\Gamma}_{tuq} C^{q}_{.rs} 2\overline{\Omega}_{m,turs} = \frac{1}{4q^2} \overline{C}_{tuq} \overline{C}^{q}_{.rs}$

## 5.3 The Lagrangian.

In this section we calculate the first two terms of the Langrangian (3.10) . We shall need the following identity : $(\omega^{d_1} \wedge ... \wedge \omega^{d_p}) \wedge *(\omega^{a_1} \wedge ... \wedge \omega^{a_p}) = \varepsilon\, \eta^{d_{i_1} a_1} ... \eta^{d_{i_p} a_p}\, dV$
where : $\varepsilon = \pm 1$ if the index set $\{d_i\}$ is equal to the set $\{a_i\}$ up to a permutation, and $\varepsilon = 0$ otherwise, and where : $dV = dV^n . dV^m$ . The first term of (3.10) becomes :

$$-\beta^2\, \Omega^{ab} \wedge *(\omega_a \wedge \omega_b) = -2\, \beta^2\, \Omega^{ab}_{\;\;..ab}\, dV \qquad (5.18)$$

which is, up to a factor $-2\beta^2$ , the scalar curvature of $V^{n+m}$ . Neglecting the terms of order $q$ and above in equations (5.15) , (5.16) , (5.17) , one gets :

$$\Omega^{ab}_{\;\;..ab} = \Omega_{n,\;\;..ij}^{\;\;\;ij} - \frac{m}{q} D_k q^k + 2m(m-1) \frac{q_m q^m}{q^2} + \overline{\Omega}_{m,\;\;..rs}^{\;\;\;rs} \qquad (5.19)$$

which contains the scalar curvature of the $n$ dimensional « macroscopic » space .

The second term of (3.10) is : $-\frac{1}{8} \Omega^{ab} \wedge *\Omega_{ab} = -\frac{1}{4} \Omega^{abcd} \Omega_{abcd}$

where : $\Omega^{abcd} \Omega_{abcd} = \Omega_{n,}^{\;\;\;ijkl} \Omega_{n,ijkl} + \frac{1}{16} G^{t}_{ij} G^{uij}\, \overline{C}_{trs} \overline{C}_{u}^{\;.rs} \qquad (5.20)$

$$+2\, \eta^{ij} \eta^{tu} \eta^{kl} \eta^{rs} \left( \frac{\eta_{rt}}{q} D_k q_i + \frac{1}{4} G^{p}_{.ik} \overline{C}_{ptr} \right) \left( \frac{\eta_{su}}{q} D_l q_j + \frac{1}{4} G^{q}_{.jl} \overline{C}_{qus} \right)$$
$$+ \frac{1}{16} G^{r}_{kl} G^{skl} \overline{C}_{prq} \overline{C}_{tsu} \eta^{pt} \eta^{qu} + \frac{m(m-1)}{2} \frac{q_k q^k}{q^2} \frac{q_l q^l}{q^2} + \overline{\Omega}_{m,turs} \overline{\Omega}_{m,}^{\;\;\;turs}$$

The gauge field quadratic terms are , using condition (4.28) :

$$\frac{1}{4} G^{t}_{ij} G^{uij}\, \overline{C}_{trs} \overline{C}_{u}^{\;.rs}$$

where : $\overline{C}_{trs} \overline{C}_{u}^{\;.rs} = -\overline{C}^{r}_{.ts} \overline{C}^{s}_{.ur}$ is the Killing form of the Lie algebra of the group whose $V^m$ is the parameter space.

## Conclusion.

The usual gauge field Lagrangian is a direct consequence of the Lagrangian (3.10). The Eintein-Hilbert Lagrangian is modified by the first term of (5.20). The evolution of $V^m$ , in the model considered here, depends on the evolution of the function $q$ whose Lagrangian is quartic. At the « macroscopic » level, the couplings in (4.31), (5.19), (5.20) do not depend on the function $q$ and, therefore, do not depend on time.

# 6. Mass terms, torsion constraints.

## 6.1 The operator $D_m$ .

The « macroscopic » Dirac equation (4.31), contains the operator (4.6). Using (5.12) , this operator is :

$$D_m = \frac{1}{q}\gamma^r \left( \overline{h}_r^{\varphi}\, \partial_\varphi - \overline{C}_{str}\frac{\gamma^s \gamma^t}{8} \right) \tag{6.1a}$$

We define : $D_{V^m} = D_m + S_{rst}\dfrac{\gamma^r \gamma^s \gamma^t}{4} = \gamma^r h_r^{\varphi}(\partial_\varphi + \overline{\Gamma}_{st\varphi}\dfrac{\gamma^s\gamma^t}{4}) + S_{rst}\dfrac{\gamma^r\gamma^s\gamma^t}{4}$ (6.1b)

where, in the last term, all the indices are different. We consider the scalar product of two spinors on $V^m$ : $S = \int_{V^m} \overline{\varphi}\, \psi\, dV = \int_{V^m} \varphi^+ \gamma^0 \psi\, dV$ . The calculations which lead to (2.17) can be applied to $D_{V^m}$ , then : the operator : $i\,D_{V^m}$ is hermitic on $V^m$ (6.2)

The product: $S = \int_{V^m} \overline{\varphi}\, D_{V^m} \psi\, dV$ is invariant with respect to re-definitions of the local orthonormal frames of $V^m$ .

For an isometry, the associated Lie derivative commutes with the Dirac operator $d-\delta$. It is shown at the end of this chapter that : $\left[\widehat{T}_x , D_{V^m}\right] = 0$ (6.3)

However, the operator $D_{V^m}$ does not commute with $\not{D}$ (4.31), because :

$$\left[\gamma^i h_i^{\mu}\partial_\mu , \gamma^r h_r^{\varphi}\partial_\varphi\right] = (\gamma^i\gamma^r - \gamma^r\gamma^i)\, h_i^{\mu} h_r^{\varphi}\,\partial_{\mu\varphi} - \gamma^i\gamma^r \frac{h_i^{\mu} q_\mu}{q} h_r^{\varphi}$$

where the first term on the right is not zero.

## 6.2 Mass term of the Lagrangian.

The usual Lagrangian of a spinor field is :

$$L = \tfrac{1}{2}\left(\overline{\psi}\, i \not{D} \psi + h.c.\right) \pm m \overline{\psi}\psi \tag{6.4}$$

where : $\not{D}$ is defined in (2.15), and the resulting Euler-Lagrange equations are : $i\not{D}\psi = \pm m \psi$ . As said in chapter 4, in order to obtain equation (4.2) one must « diagonalise » equation (4.31). This can be done by multiplying it by the operator (A24) : $\gamma_D$ . The equation of motion is therefore : $i\gamma_D \not{D}\psi = \pm m \gamma_D \psi$ .

We define : $$\not{D}_D = \gamma_D \not{D} \tag{6.5}$$

we would like to have: $$i\not{D}_D \psi = m\psi \tag{6.6}$$

To obtain this equation, the mass term of the Lagrangian (6.4) can be replaced by :

$$m\overline{\psi}\gamma_D\psi$$

since, from (A26) : $(\gamma_D)^2 = I_{q_{n+m}}$ , and, with (A26) et (A27) :

$$\left(\overline{\psi}\gamma_D\psi\right)^+ = \psi^+ \gamma_D{}^+ \gamma^{0+}\psi = \psi^+\gamma_D\gamma^0\psi = \psi^+\gamma^0\gamma_D\psi = \overline{\psi}\gamma_D\psi$$

In conclusion, the usual Dirac equation can be obtain from the following Lagrangian :

$$L=\tfrac{1}{2}\left(\overline{\psi}\, i\not{D}\psi+h.c.\right)\pm m\overline{\psi}\,\gamma_D\psi \tag{6.7}$$

With (6.7) and (4.31), the « macroscopic » Dirac equation, in which the terms of order $O(q)$ or less have been neglected, is

$$\gamma_D\not{D}=\gamma_D\gamma^i h_i^\mu\left(\partial_\mu+\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}+A_\mu^t\widehat{T}_t\right)\psi+\gamma_D D_m\psi+\gamma_D S_{abc}\frac{\gamma^a\gamma^b\gamma^c}{4}=\pm i\, m\psi \tag{6.8}$$

## 6.3 The operator $D_m$ in the « macroscopic » Dirac equation .

The term $\gamma_D D_m\psi$ in the equation (6.8) is not negligible (see (6.1a)). The resolution of (6.8) would be greatly simplified if this term would commute with the others. One has :

$$\left[\gamma_D\gamma^i h_i^\mu\partial_\mu\ ,\ \gamma_D D_{V^m}\right]=\gamma^i h_i^\mu\left(\partial_\mu D_{V^m}-D_{V^m}\partial_\mu\right)=-\frac{\gamma^i h_i^\mu q_\mu}{q}D_{V^m}$$

The next term to calculate is : $\left[\gamma_D\gamma^i h_i^\mu\,\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}\ ,\ \gamma_D D_{V^m}\right]$ , but : $\overline{\Gamma}_{jk\mu}=h_\mu^i\overline{\Gamma}_{jki}+O(q^2)$ depends only on the $V^n$ coordinates at the considered order, then, using (A27) and (A1) :

$$\left[\gamma_D\gamma^i h_i^\mu\,\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}\ ,\ \gamma_D D_{V^m}\right]\approx h_i^\mu\,\overline{\Gamma}_{jk\mu}\left[\gamma_D\gamma^i\frac{\gamma^j\gamma^k}{4}\ ,\ \gamma_D D_{V^m}\right]=0$$

Now, with (4.27) , (4.29) and (6.3) :

$$\left[\gamma_D\gamma^i h_i^\mu A_\mu^t\widehat{T}_t\ ,\ \gamma_D D_{V^m}\right]=\gamma^i A_i^t\widehat{T}_t D_{V^m}+D_{V^m}\gamma^i A_i^t\widehat{T}_t=\gamma^i A_i^t\left[\widehat{T}_t\ ,D_{V^m}\right]=0$$

Finally, if $q_\mu=0$ or if $q_\mu\ll O(q)$ :

$$\left[\gamma_D\gamma^i h_i^\mu\left(\partial_\mu+\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}+A_\mu^t\widehat{T}_t\right)\ ,\ \gamma_D D_{V^m}\right]\approx 0 \tag{6.9}$$

If the torsion is null, the term $\gamma_D D_{V^m}$ commutes with the other terms of (6.8), and one can find a common set of eigenvectors to $\gamma_D D_{V^m}$ and to : $\gamma_D\gamma^i h_i^\mu\left(\partial_\mu+\overline{\Gamma}_{jk\mu}\frac{\gamma^j\gamma^k}{4}+A_\mu^t\widehat{T}_t\right)$ .

The eigenvalues of : $D_{V^m}$ are real and that those of : $\gamma_D D_{V^m}$ are imaginary. Consider the product : $(D_m\psi)^+(D_m\psi)=(\lambda^+\lambda)(\psi^+\psi)$ where :

$$(D_m\psi)^+(D_m\psi)=-\partial_\varphi(\psi^+\gamma^r h_r^\varphi(D_m\psi))+\psi^+\gamma^r\partial_\varphi(h_r^\varphi)(D_m\psi)+\psi^+\gamma^r h_r^\varphi\partial_\varphi(D_m\psi)$$
$$+\psi^+(\gamma^r h_r^\varphi\,\overline{\Gamma}_{st\varphi}\frac{\gamma^s\gamma^t}{4}-\overline{\Gamma}^r_{.sr}\gamma^s)(D_m\psi)$$

From (1.15g) and (4.28) : $\overline{\Gamma}^s_{.rs}=-C^s_{.rs}$ , and using (1.21) , one gets :

$$(D_m\psi)^+(D_m\psi)\sqrt{g_{V^m}}=-\partial_\varphi(\psi^+\gamma^r h_r^\varphi(D_m\psi)\sqrt{g_{V^m}})+\psi^+D_m(D_m\psi)\sqrt{g_{V^m}}$$

This calculation can be directly applied to $D_{V^m}$ , and since in (6.1b) the indices $r,s,t$ of the torsion term are all different, one has :

$$(D_{V^m}\psi)^+(D_{V^m}\psi)\sqrt{g_{V^m}}=-\partial_\varphi(\psi^+\gamma^r h_r^\varphi(D_{V^m}\psi)\sqrt{g_{V^m}})+\psi^+D_{V^m}(D_{V^m}\psi)\sqrt{g_{V^m}} \tag{6.10}$$

Now let us assume that $\psi$ is an eigenvector of $D_{V^m}$ : $D_{V^m}\psi=\lambda\psi$

If $V^m$ is compact, after integrating on $V^m$ , one has : $\lambda^2 = \lambda^+ \lambda = |\lambda|^2 > 0$ and then : $\lambda$ is real. That same calculations can be done for $\gamma_D D_{V^m}$ , and in that case, its eigenvalues are imaginary.

When the torsion is null the term : $\gamma_D D_{V^m}$ commutes with the other terms of (6.8), and one can find a common set of eigenvectors to these operators. The operator $\gamma_D D_{V^m}$ contributes to the mass term at the order $1/q$ , which is very large, except, of course, in the case of the zero eigenvalue. If the torsion is not zero, what are the conditions for keeping the commutator (6.9) null ? Can the torsion terms be considered as a perturbation?

## 6.4 What can be said about torsion ?

The torsion has been introduced in chapter 3 but it can be considered as an external field whose dynamics is not completely defined by (3.10b). In the Dirac equation it is not necessary to require that the operator $D_m$ commutes with the torsion terms. They can be treated as a perturbation if its intensity allows this.

In this section we look at which conditions the term $\gamma_D D_{V^m}$ commutes with the torsion terms of (6.8) :

$$\left[ \gamma_D D_{V^m} \ , \ \gamma_D S_{abc} \frac{\gamma^a \gamma^b \gamma^c}{4} \right] = -\gamma^r \partial_r S_{abc} \frac{\gamma^{abc}}{4} - \frac{1}{4} S_{abc} \left( D_m \gamma^{abc} + \varepsilon \gamma^{abc} D_m \right)$$

$$- \frac{1}{16} S_{abc} S_{rst} \left( \gamma^{rst} \gamma^{abc} + \varepsilon \gamma^{abc} \gamma^{rst} \right) \qquad (6.11)$$

where : $\partial_r = h_r^\varphi \partial_\varphi$ and : $\gamma^{abc} = \gamma^a \gamma^b \gamma^c$ for : $a \neq b \neq c \neq a$ and where : $\varepsilon = \pm 1$ , according to : $\gamma_D \gamma^{abc} \gamma_D = \varepsilon \gamma^{abc}$ .

This section has been added for completeness, the result is that the torsion must be negligible, its reading can be skipped.

In equation (6.11), the first term on the right is constrained by (1.13). At chapter 4 , $V^m$ has been assumed to be be the parameter space of $G$ . From the Maurer-Cartan equations, one can chose : $$S_{rst} = 0 \quad \leftrightarrow \quad S_{\chi\tau\varphi} = 0 \qquad (6.12)$$

and the last term of (6.11) is zero.

In order to compute the constraint (1.13) we go back to the definitions of chapter 1. We can take local frames such that : $$\overline{h}_r^\varphi = X_r^\varphi \qquad (6.13)$$

Then : $q_{ra.}{}^b \, h_b = \left[ \overline{h}_r , h_a \right]$ , and for : $a = s$ , using (1.16a) :

$$q_{rs.}{}^b \, h_b = \left[ \overline{h}_r , h_s \right] = \left[ \overline{h}_r , \frac{1}{q} \overline{h}_s \right] = \frac{1}{q} \overline{C}^t_{.rs} \overline{h}_t$$

therefore (see also chapter 4) : $$q_{rs.}{}^t = \overline{C}^t_{.rs} \qquad (6.14a)$$

The other components of : $q_{xa.}{}^b$ are , using (1.12b) and (1.12c) :

$$q_{ri.}{}^j = q_{rs.}{}^j = 0 \qquad (6.14b)$$

At last, using (4.29) in (1.12d) : $$q_{ri.}{}^t = q A_i^s \, \overline{C}^t_{.rs} \qquad (6.14c)$$

With these relations, the constraint (1.13) is :

$$q\partial_r S_{abc} = q h_r^{\varphi}\,\partial_\varphi S_{abc} = \overline{h}_r^{\varphi}\,\partial_\varphi S_{abc} = -S_{sbc}\,q_{r.a}^{\;\;s} - S_{ibc}\,q_{r.a}^{\;\;i} + S_{asc}\,q_{rb.}^{\;\;s} + S_{abs}\,q_{rc.}^{\;\;s} \tag{6.15}$$

We have now all the ingredients to compute the commutator (6.11). When the indices $a,b,c$ are of the type $i,j,k$ , $\varepsilon = 1$ and the commutator (6.11) reduces to : $-\gamma^r\,\partial_r S_{ijk}\dfrac{\gamma^{ijk}}{4}$ , with : $i \neq j \neq k \neq i$ , where : $\partial_r S_{ijk} = (S_{ijs}\,A_k^t - S_{iks}\,A_j^t)\,\overline{C}_{.rt}^{\;s}$

The commutator (6.11) can be zero, for this set of indices, if : $S_{ijs}\gamma^{ij} = 0$ , that is to say, if : $S_{ijs}$ is symmetric with respect to the two first indices, or if : $S_{ijs} = 0$ . It can be negligible if :

$$S_{ijs} \leq O(q) \tag{6. 16}$$

If the indices $a,b,c$ are of the type : $i,j,p$ : $\varepsilon = -1$ , and with the relation (6.16), it remains to consider the case $S_{pij}$ . In that case :

$$\left[\,\gamma_D\,D_{V^m}\ ,\ \gamma_D\,S_{pij}\frac{\gamma^{pij}}{4}\right] = -\gamma^r\,\partial_r S_{pij}\frac{\gamma^{pij}}{4} + \frac{1}{4}\,S_{pij}\,\gamma^{ij}\left[\gamma^p, D_m\right] \tag{6.17}$$

$$= -\frac{\gamma^r\gamma^{ij}}{4}\left\{\partial_r S_{pij}\,\gamma^p + S_{pij}\,\overline{\Gamma}_{s.r}^{\;p}\,\gamma^s\right\} + \ S_{pij}\frac{\gamma^{ij}}{4}\left[\gamma^p,\gamma^r\right]D_r$$

where : $D_r = (\partial_r + \Gamma_{str}\dfrac{\gamma^{st}}{4})$ , and :

$$q\partial_r S_{pij} = -S_{sij}\,q_{r.p}^{\;\;s} - S_{mij}\,q_{r.p}^{\;\;m} + S_{psj}\,q_{ri.}^{\;\;s} + S_{pis}\,q_{rj.}^{\;\;s}$$

$$= -S_{sij}\,\overline{C}_{pr.}^{\;\;s} - S_{.ij}^{m}\,q\,A_m^s\,\overline{C}_{prs} + S_{psj}\,q\,A_i^q\,\overline{C}_{.rq}^{\;s} + S_{pis}\ q\,A_j^q\,\overline{C}_{.rq}^{\;s}$$

Finally the commutator (6.17) is :

$$= \frac{\gamma^r\gamma^{ij}}{8q}\ \gamma^p\,S_{sij}\,\overline{C}_{pr.}^{\;\;s} + \frac{\gamma^r\gamma^{ij}}{4}\left(S_{.ij}^{m}\,A_m^s\,\overline{C}_{prs} - S_{psj}\,A_i^q\,\overline{C}_{.rq}^{\;s} - S_{pis}\ \ A_j^q\,\overline{C}_{.rq}^{\;s}\right) + \ S_{pij}\frac{\gamma^{ij}}{4}\left[\gamma^p,\gamma^r\right]D_r$$

it will be negligible if :

$$S_{pij} \ll O(q) \qquad , \qquad S_{mij} \ll 1 \qquad , \qquad S_{psj} \ll 1 \tag{6.18}$$

When the indices $a,b,c$ are of the type : $i,p,q$ : $\varepsilon = 1$ . The relation (6.11) is in that case :

$$\left[\,\gamma_D\,D_{V^m}\ ,\ \gamma_D\,S_{ipq}\,\gamma^{ipq}\right] = -\gamma^r\,\partial_r S_{ipq}\,\gamma^{ipq} + \ S_{ipq}\,\gamma^i\left[D_m\ ,\gamma^{pq}\right] \tag{6.19a}$$

$$= -\gamma^r\,\partial_r S_{ipq}\,\gamma^{ipq} + \gamma^i\,S_{ipq}\,\gamma^r\left[\gamma^{st}\ ,\gamma^{pq}\right]\frac{\overline{\Gamma}_{str}}{4} + \ S_{ipq}\,\gamma^i\left[\gamma^r\ ,\gamma^{pq}\right]D_r$$

and likewise : $\left[\,\gamma_D\,D_{V^m}\ ,\ \gamma_D\,S_{pqi}\,\gamma^{pqi}\right] =$

$$= -\gamma^r\,\partial_r S_{pqi}\,\gamma^{pqi} + \gamma^i\,S_{pqi}\,\gamma^r\left[\gamma^{st},\gamma^{pq}\right]\frac{\overline{\Gamma}_{str}}{4} + S_{pqi}\,\gamma^i\left[\gamma^r,\gamma^{pq}\right]D_r \tag{6.19b}$$

The relation (6.15) is : $q\partial_r S_{ipq} = S_{isq}\,\overline{C}_{.rp}^{\;s} + S_{ips}\,\overline{C}_{.rq}^{\;s}$

and the first term of (6.19a) is : $-\gamma^r\,\partial_r S_{ipq}\,\gamma^{ipq} = -\dfrac{2}{q}\,S_{isq}\,\gamma^r\,\overline{C}_{.rp}^{\;s}\,\gamma^{ipq}$

In equation (6.19a) only the torsion terms of the type : $S_{ipq}$ contribute, then, (6.19a) is macroscopically negligible if :

$$S_{ipq} \ll O(q) \tag{6.20a}$$

The case $a,b,c = p,q,i$ is treated in the same way, one has :

$$q\,\partial_r S_{pqi} = -S_{sqi}\,\overline{C}_{pr.}^{\;\;\;\;s} - q\,S^{m}_{.qi}\,A^{t}_{m}\,\overline{C}_{prt} + S_{psi}\,\overline{C}^{s}_{.rq}$$

and, like in the former case, one must have : $\quad S_{pqi} \ll O(q)$ (6.20b)

These constraints can be written with coordinate indices. The above conditions are satisfied if : $S_{\nu\mu\varphi} = S_{\mu\nu\varphi}$ or if : $S_{\mu\nu\varphi} \ll O(q)$ , $S_{\nu\tau\varphi} = S_{\tau\nu\varphi}$ or : $S_{\mu\tau\varphi} \ll O(q)$
the other constraints are : $S_{\varphi\tau\mu} = 0$ , $S_{\mu\varphi\tau} = 0$ , $S_{\varphi\mu\nu} = 0$ , $S_{\mu\nu\rho} \ll 1$

## Annex : calculation of (6.3).

Here we calculate the commutator (6.3) and we begin with the case of zero torsion. Using (2.9) : $\quad \left[\widehat{T}_x , D_m\right] = \left[X_x - \frac{1}{4} q_{xst}\,\gamma^s\gamma^t \;,\; \gamma^r h_r - \frac{1}{8} C_{ruv}\,\gamma^r\gamma^u\gamma^v - \frac{1}{2} C^{s}_{.rs}\,\gamma^r\right]$

where we have written : $h_r = h_r^{\varphi}\,\partial_\varphi$ , and where : $r \neq u \neq v \neq r$ . In order to insist on the fact that the indices in a product of $\gamma$ matrices are all différent, we write, for instance : $\gamma^{ruv}$ .
Developing :

$$\left[\widehat{T}_x , D_m\right] = q_{xr.}^{\;\;\;\;t}\,h_t\,\gamma^r - \frac{1}{8} X_x(C_{ruv})\,\gamma^{ruv} - \frac{1}{2} X_x(C^{s}_{.rs})\,\gamma^r + \frac{1}{4} h_r(q_{xst})\,\gamma^{rst} + \frac{1}{4} q_{xst}\left[\gamma^r , \gamma^{st}\right] h_r$$
$$+ \;\frac{1}{32}\, q_{xst}\, C_{ruv}\left[\gamma^{st}, \gamma^{ruv}\right] + \frac{1}{8}\, q_{xst}\, C^{u}_{.ru}\left[\gamma^{st}, \gamma^{r}\right]$$

The term : $X_x(C_{ruv})$ is obtained from : $\left[X_x , \left[h_u , h_v\right]\right]$ , using Jacobi's identities :

$$\left[X_x , \left[h_u , h_v\right]\right] = \left[X_x , C^{r}_{.uv}\, h_r\right] = q_{xr.}^{\;\;\;\;t}\, C^{r}_{.uv}\, h_t + X_x(C^{r}_{.uv})\, h_r$$
$$= -\left[h_v , \left[X_x , h_u\right]\right] + \left[h_u , \left[X_x , h_v\right]\right] = -\left[h_v , q_{xu.}^{\;\;\;\;t}\, h_t\right] + \left[h_u , q_{xv.}^{\;\;\;\;t}\, h_t\right]$$
$$= -q_{xu.}^{\;\;\;\;t}\, C^{s}_{.vt}\; h_s + q_{xv.}^{\;\;\;\;t}\, C^{s}_{.ut}\; h_s - h_v(q_{xu.}^{\;\;\;\;t})\, h_t + h_u(q_{xv.}^{\;\;\;\;t})\, h_t$$

and using (1.11) : $\quad X_x(C^{s}_{.rs}) + h_s(q_{xr.}^{\;\;\;\;s}) = q_{xr.}^{\;\;\;\;t}\, C^{s}_{.ts}$

Finally : $\quad \left[\widehat{T}_x , D_m\right] = \frac{1}{8}\left(q_{xsr}\, C^{s}_{.uv} + 2\, q_{xu.}^{\;\;\;\;t}\, C_{rvt}\right)\gamma^{ruv} + \frac{1}{8}\, q_{xst}\, C_{ruv}\left[\gamma^{st}, \gamma^{ruv}\right]$

which gives, when the torsion is null : $\left[\widehat{T}_x , D_m\right] = 0$

To include the torsion one has to calculate :

$$\left[\widehat{T}_x , D_{V^m}\right] = \left[\widehat{T}_x \;,\; \gamma^r h_r - \frac{1}{8} C_{rst}\,\gamma^{rst} + \frac{1}{4} S_{rst}\,\gamma^{rst} - \frac{1}{2} C^{s}_{.rs}\,\gamma^r\right]$$

with respect to the former calculation one must calculate :

$$\left[\widehat{T}_x , S_{rst}\,\gamma^{rst}\right] = X_x(S_{rst})\,\gamma^{rst} - \frac{1}{4} q_{xst}\, S_{ruv}\left[\gamma^{st}, \gamma^{ruv}\right]$$

which gives : $\quad \left[\widehat{T}_x , S_{rst}\,\gamma^{rst}\right] = \left(X_x(S_{rst}) + q_{xur}\, S^{u}_{.st} - q_{xt.}^{\;\;\;\;u}\, S_{rsu} - q_{xs.}^{\;\;\;\;u}\, S_{rut}\right)\gamma^{rst}$

and finally, with condition (1.13) :

$$\left[\widehat{T}_x , D_{V^m}\right] = 0$$

## Summary.

If the torsion is zero or if it can be considered as a perturbation, the commutation relation (6.9) help to solve the Dirac equation by looking for a system of common eigenvectors for

$D_{V^m}$ and the « macroscopic » Dirac operator. The eigenvalues of $D_{V^m}$ contribute to the mass term.

## Appendix A. Representation of the Dirac matrices.

In this appendix the space-time possesses orthonormal Cartesian coordinates and the metric $\eta^{ab}$ is diagonal with : $\eta^{aa} = \pm 1$ .

### A.1 The Dirac matrices.

The dimension of a spinor in $\mathbb{R}^n$ is : $q_n = 2^{[\frac{n}{2}]}$ where the brackets mean : integer part of $n/2$ . The Dirac matrices are defined by the constraints :

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\, \eta^{ab}\, I_{q_n} \qquad \text{(A1)}$$

where $I_{q_n}$ is the unit matrix of dimension : $q_n \times q_n = 2^{[\frac{n}{2}]} \times 2^{[\frac{n}{2}]}$ .

Recall that the Pauli matrices are :

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(A2)}$$

which satisfy the relations :

$$\sigma^i \sigma^j + \sigma^j \sigma^i = 0 \ \text{ si } i \neq j \ , \ (\sigma^i)^2 = I \ , \ \sigma^i \sigma^j = i\, \overline{\varepsilon}^{ijk}\, \sigma^k \ , \ (\sigma^i)^+ = \sigma^i$$

We assume that we already know a representation of the Dirac matrices in the 4-dimensional Minkowski space $M^4$ and we shall build a set of satisfying (A1) by iteration.

Let : $A, B, C$ be matrices of dimension $2^{[\frac{n}{2}]} \times 2^{[\frac{n}{2}]}$ . We set :

$$A_+ = \sigma^i \otimes A \qquad , \qquad B_+ = \sigma^j \otimes B \qquad \text{(A3)}$$

for instance :

$$A_+ = \sigma^1 \otimes A = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \qquad B_+ = \sigma^2 \otimes B = \begin{pmatrix} 0 & -iB \\ iB & 0 \end{pmatrix} \qquad C_+ = \sigma^3 \otimes C = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \quad \text{( A4)}$$

In the following, since we consider only square matrices, we shall simply say that their dimension is : $2^{[\frac{n}{2}]}$ . By direct calculation one has :

$$\begin{aligned} (\sigma^i \otimes A)(\sigma^j \otimes B) &= (\sigma^i \sigma^j) \otimes (AB) && \text{(A5)} \\ &= i\, \overline{\varepsilon}^{ijk}\, \sigma^k \otimes (AB) && \text{si : } i \neq j \\ &= I_2 \otimes (AB) && \text{si : } i = j \end{aligned}$$

where : $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

The (anti)-commutators are then :

$$A_+ B_+ \pm B_+ A_+ == i\,\overline{\varepsilon}^{ijk}\,\sigma^k \otimes (AB \mp BA) \qquad \text{si :} \quad i \neq j \qquad \text{(A6)}$$
$$A_+ B_+ \pm B_+ A_+ == I_2 \otimes (AB \pm BA) \qquad \text{si :} \quad i = j$$

and, by direct calculation :

$$(\sigma^i \otimes A)^+ = \sigma^i \otimes A^+ \qquad \text{(A7)}$$

We can now use these results to build the Dirac matrices in $M^{n+2}$ .

We name : $\tau^1$ one of the Pauli matrices. For instance $\tau^1 = \sigma^2$ . We call : $\gamma^a$ $(0 \le a < n)$ the Dirac matrices in $M^n$ , and set :

$$\gamma_+^a = \tau^1 \otimes \gamma^a \quad , \quad \gamma_+^n = \tau^2 \otimes b \quad , \quad \gamma_+^{n+1} = \tau^3 \otimes c \qquad \text{(A8)}$$

where : $b$ and $c$ are matrices to be determined. Using (A6), the basic constraint (A1) is satisfied : $\gamma_+^a \gamma_+^b + \gamma_+^b \gamma_+^a = 2\,\eta^{ab} I_{q_{n+2}}$

One must have also : $(\gamma_+^n)^2 = \eta^{nn} I_{q_{n+2}}$ and : $\gamma_+^a \gamma_+^n + \gamma_+^n \gamma_+^a = 0$ , that is to say :

$$(\gamma_+^n)^2 = I_2 \otimes b^2 = \eta^{nn} I_{q_{n+2}} \quad \text{and then : } b^2 = \eta^{nn} I_{q_n}$$

and : $\gamma_+^a \gamma_+^n + \gamma_+^n \gamma_+^a = \pm\, i\, \tau^3 \otimes (\gamma^a b - b\gamma^a) = 0$

The conditions are the same for : $\gamma_+^{n+1}$ . A possible solution is :

$$b = \pm\sqrt{\eta^{nn}}\ I_{q_n} \qquad \text{et :} \qquad c = \pm\sqrt{\eta^{nn}}\ I_{q_n} \qquad \text{(A9)}$$

It remains to check that : $\gamma_+^n \gamma_+^{n+1} + \gamma_+^{n+1} \gamma_+^n = 0$ which is a direct consequence of (A6) and (A9) . Finally, (A8) et (A9) are solutions of :

$$\gamma_+^a \gamma_+^b + \gamma_+^b \gamma_+^a = 2\,\eta^{ab} I_{q_{n+2}} \qquad \text{for :} \quad 0 \le a < n+2 \qquad \text{(A10)}$$

## A.2 Dirac matrices Hermiticity and other properties , first case.

In this section we consider metrics of the type : $(+\ -\ -\ \ldots\ -)$ where : $\eta^{aa} = -1$ si : $a > 0$ .

One wants to check that the Dirac matrices satisfy the Hermiticity properties :

$$(\gamma^0)^+ = \gamma^0 \quad , \quad \gamma^0 (\gamma^a)^+ \gamma^0 = \gamma^a$$

With these constraint the hamiltonian of a free particle is Hermitic, and the probabilities are conserved (next section).

With (A7) one has directly :

$$(\gamma_+^0)^+ = \gamma_+^0 \qquad \text{(A11a)}$$

For $0 < i < n$ , one has : $\gamma_+^0 (\gamma_+^i)^+ - (\gamma_+^i)\gamma_+^0 = I_2 \otimes (\gamma^0 \gamma^{i+} - \gamma^i \gamma^0) = 0$ (A11b)

and for the other Dirac matrices :

$$\gamma_+^0 (\gamma_+^n)^+ - (\gamma_+^n)\gamma_+^0 = (\tau^1 \otimes \gamma^0)(\tau^2 \otimes (\mp i I_{q_n})) - (\tau^2 \otimes (\pm i I_{q_n}))(\tau^1 \otimes \gamma^0)$$

and with (A6) : $\gamma_+^0 (\gamma_+^n)^+ - (\gamma_+^n)\gamma_+^0 = 0$

The calculation is the same for : $\gamma_+^{n+1}$ , then :

$$\gamma_+^0 (\gamma_+^a)^+ = (\gamma_+^a)\gamma_+^0 \qquad \text{whatever :} \quad 0 \le a < n+2 \qquad \text{(A12)}$$

Given the Dirac matrices for $n$ dimensional space-time we have build a set of matrices satisfying the conditions (A1) , (A11) and (A12) for the dimension $n+2$ . The next step is to get explicit Dirac matrices expressions for the dimension $n+m$ ($m$ even) .

For any matrix $A$ we have : $\sigma^i \otimes (\sigma^j \otimes A) = (\sigma^i \otimes \sigma^j) \otimes A$ (A13)

The Dirac matrices of $M^k$ will be written $\gamma_k^a$ and let : $p = m/2$ .

We have : $\gamma_{n+m}^a \sim A_{n+m} = \sigma^{i_1} \otimes (\sigma^{i_2} \otimes ( \ldots \otimes (\sigma^{i_p} \otimes A_n) \ldots ))$

Likewise, let us define : $B_{n+m} = \sigma^{j_1} \otimes (\sigma^{j_2} \otimes ( \ldots \otimes (\sigma^{j_p} \otimes B_n) \ldots ))$

Using (A13) oe can write :

$$A_{n+m} B_{n+m} = (\sigma^{i_1} \sigma^{j_1}) \otimes (\sigma^{i_2} \sigma^{j_2}) \otimes \ldots \otimes (\sigma^{i_p} \sigma^{j_p}) \otimes (A_n B_n) \qquad \text{(A14)}$$

With (A5) the following commutator :

$$\left[\sigma^i \otimes A , \sigma^j \otimes B\right] = (\sigma^i \sigma^j) \otimes (AB) - (\sigma^j \sigma^i) \otimes (BA)$$
$$= (\sigma^i \sigma^j - \sigma^j \sigma^i) \otimes (AB) + (\sigma^j \sigma^i) \otimes (AB - BA)$$

is : $$\left[\sigma^i \otimes A , \sigma^j \otimes B\right] = \left[\sigma^i , \sigma^j\right] \otimes (AB) + (\sigma^j \sigma^i) \otimes \left[A, B\right] \qquad \text{(A15)}$$

which can be used by iteration .

Let us consider the matrix : $$\hat{\gamma} = \gamma^{r_1} \gamma^{r_2} \ldots \gamma^{r_m} \qquad \text{(A16)}$$

which is equivalent , in $V^m$ , to the matrix $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ of $M^4$ .

$\hat{\gamma}$ satisfies : $\left[\hat{\gamma} , \gamma^s\right] = 0$ if $m$ is odd , $\hat{\gamma} \gamma^s + \gamma^s \hat{\gamma} = 0$ if $m$ is even (A17)

then : $$\left[\hat{\gamma} , \gamma^r \gamma^s\right] = 0 \qquad \text{(A18)}$$

and (see (2.9)) : $$\left[\hat{\gamma} , \widehat{M}_x\right] = 0 \qquad \text{(A19)}$$

With the metric used in this section : $$(\hat{\gamma})^2 = (-1)^{m+1} I_{q_{n+m}} \qquad \text{(A20)}$$

and : $$(\hat{\gamma})^+ = (-1)^{\frac{m(m+1)}{2}} \hat{\gamma} \qquad \text{(A21)}$$

Let us consider the case $m = 2$ . The Dirac matrices are (A8) :

$$\gamma_{n+2}^a = \tau^1 \otimes \gamma^a \quad \text{if :} \quad 0 \le a < n \quad , \quad \gamma_{n+2}^n \sim \tau^2 \otimes I_{q_n} \quad , \quad \gamma_{n+2}^{n+1} \sim \tau^3 \otimes I_{q_n}$$

Now, for : $m = 4$ , the matrices of the type $\gamma^r$ are (up to a factor $\pm i$ ) :

$$\gamma_{n+4}^n \sim \tau^1 \otimes \tau^2 \otimes I_{q_n} \quad , \quad \gamma_{n+4}^{n+1} \sim \tau^1 \otimes \tau^3 \otimes I_{q_n} \quad , \quad \gamma_{n+4}^{n+2} \sim \tau^2 \otimes I_{q_{n+2}} \quad , \quad \gamma_{n+4}^{n+3} \sim \tau^3 \otimes I_{q_{n+2}}$$

then, with (A5) : $$\hat{\gamma}_{n+4} \sim \tau^1 \otimes \tau^1 \otimes I_{q_{n+4}}$$

More generally, knowing the Dirac matrices for the dimension $n + m$ , the matrices of type $\gamma^r$ for the dimension $n + m + 2$ are :

$$\gamma_{n+m+2}^{r_i} = \tau^1 \otimes \gamma_{n+m}^{r_i} \quad \text{pour : } 1 \le i \le m \quad , \quad \gamma_{n+m+2}^{n+m} \sim \tau^2 \otimes I_{q_{n+m}} \quad , \quad \gamma_{n+m+2}^{n+m+1} \sim \tau^3 \otimes I_{q_{n+m}}$$

the product of the $m$ first matrices is :

$$\gamma_{n+m+2}^{r_1} \ldots \gamma_{n+m+2}^{r_m} = (\tau^1 \otimes \gamma_{n+m}^{r_1}) \ldots (\tau^1 \otimes \gamma_{n+m}^{r_m})$$
$$= (I_2 \otimes \gamma_{n+m}^{r_1} \gamma_{n+m}^{r_2}) (\tau^1 \otimes \gamma_{n+m}^{r_3}) \ldots (\tau^1 \otimes \gamma_{n+m}^{r_m})$$
$$= (\tau^1 \otimes \gamma_{n+m}^{r_1} \gamma_{n+m}^{r_2} \gamma_{n+m}^{r_3}) (\tau^1 \otimes \gamma_{n+m}^{r_4}) \ldots (\tau^1 \otimes \gamma_{n+m}^{r_m})$$

We assume $m$ even , then : $\gamma_{n+m+2}^{r_1} \ldots \gamma_{n+m+2}^{r_m} = I_2 \otimes \hat{\gamma}_{n+m}$

and : $$\hat{\gamma}_{n+m+2} = \pm (I_2 \otimes \hat{\gamma}_{n+m}) (\tau^2 \otimes i I_{q_{n+m}}) (\tau^3 \otimes i I_{q_{n+m}}) = \pm (I_2 \tau^2 \tau^3) \otimes \hat{\gamma}_{n+m}$$

Then : $$\hat{\gamma}_{n+m+2} = \pm i \tau^1 \otimes \hat{\gamma}_{n+m} \qquad \text{(A22)}$$

The sign depends on the choices in (A9) .

We define : $p = m/2$ when $m$ is even , then :

(if $m$ is even) : $$\hat{\gamma}_{n+m} \sim \underbrace{\tau^1 \otimes \tau^1 \otimes ... \otimes \tau^1}_{p\ fois} \otimes I_{q_n} \qquad \text{(A23)}$$

We define : $$\gamma_D = \underbrace{\tau^1 \otimes \tau^1 \otimes ... \otimes \tau^1}_{p\ fois} \otimes I_{q_n} \qquad \text{(A24)}$$

The important point is that $\gamma_D$ « diagonalises » the matrices $\gamma^i_{n+m}$ . Using (A5) again :

$$\gamma_D \gamma^i_{n+m} = \left(\tau^1 \otimes \tau^1 \otimes ... \otimes \tau^1 \otimes I_{q_n}\right)\left(\tau^1 \otimes \tau^1 \otimes ... \otimes \tau^1 \otimes \gamma^i_n\right)$$
$$= I_2 \otimes I_2 \otimes ... \otimes I_2 \otimes (I_{q_n} \gamma^i_n) = I_{q_m} \otimes \gamma^i_n \qquad \text{(A25)}$$

From (A5) and (A7) we get respectively : $\left(\gamma_D\right)^2 = I_{q_{n+m}}$ and : $\gamma_D{}^+ = \gamma_D$ (A26)

and from (A25) one gets :

$$\left[\hat{\gamma}_{n+m}, \gamma^i_{n+m}\right] = 0 \qquad \text{(A27)}$$

The commutation relations with the matrices of type $\gamma^r_{n+m}$ are given by (A17) .

## A.3 Hermiticity of the Dirac matrices , case 2 .

In this section we consider more general metrics than in section 2, where some of the metric (diagonal) elements are positive, like for the time coordinate : $\eta^{a\,a} = +1$ pour some $a > 3$ .

Let us assume that the $\gamma^a$ matrices were known for the dimension $n$ even, and that the metric for $M^n$ is the one of section 2, and let us assume that : $\eta^{n\,n} = +1$ .

From (A7) and (A9) we have : $(\gamma^n_+)^+ = \gamma^n_+$ (A28)

and, with (A11) and (A12) : $(\gamma^0_+ \gamma^n_+)^+ = -\gamma^0_+ \gamma^n_+$ , $(\gamma^n_+ \gamma^i_+)^+ = \gamma^n_+ \gamma^i_+$ for : $0 < i < n$ (A29)

These relations are of course different from the constraints (A11) et (A12) . In the case of the Minkowski space , the constraints (A11) et (A12) are linked to the conservation of the probabilities. Let us consider a spinor scalar product of the type : $S = \psi^+ M \psi$ where : $M$ is an, a priori, unknown matrix not depending on the coordinates . We have :

$$\partial_t S = \partial_t \psi^+ M \psi + \psi^+ M \partial_t \psi$$
$$= (-\alpha^i_+ \partial_i \psi - \gamma^0_+ \gamma^n_+ \partial_n \psi)^+ M \psi + \psi^+ M (-\alpha^i_+ \partial_i \psi - \gamma^0_+ \gamma^n_+ \partial_n \psi)$$

where : $\alpha^i_+ = \gamma^0_+ \gamma^i_+$ for : $i \neq 0, n$ .

$$\partial_t S = \left(-\partial_i \psi^+ (\alpha^i_+)^+ M \psi - \psi^+ M \alpha^i_+ \partial_i \psi\right) + \left(\partial_n \psi^+ \gamma^0_+ \gamma^n_+ M \psi - \psi^+ M \gamma^0_+ \gamma^n_+ \partial_n \psi\right)$$

In the case of the Minkowski space , the second parenthesis does not exist, and the probability conservation is ensured if : $(\alpha^i_+)^+ = \alpha^i_+$ and if : $M = I$ . With the condition (A28) :

$$\partial_t S = \left(-\partial_i \psi^+ \alpha^i_+ M \psi - \psi^+ M \alpha^i_+ \partial_i \psi\right) + \left(\partial_n \psi^+ \gamma^0_+ \gamma^n_+ M \psi - \psi^+ M \gamma^0_+ \gamma^n_+ \partial_n \psi\right) \qquad \text{(A30)}$$

and assuming that the relations (A29) are satisfied :

$$\partial_n S = \left(-\partial_i \psi^+ (\gamma^n_+ \gamma^i_+) M \psi - \psi^+ M (\gamma^n_+ \gamma^i_+) \partial_i \psi\right) + \left(\partial_t \psi^+ \gamma^n_+ \gamma^0_+ M \psi - \psi^+ M \gamma^n_+ \gamma^0_+ \partial_t \psi\right) \quad \text{(A31)}$$

The scalar product $S$ is conserved if : $\alpha^i_+ M = M \alpha^i_+$ , and if :
$\gamma^0_+ \gamma^n_+ M = -M \gamma^0_+ \gamma^n_+$ . This is possible if : $M \sim \gamma^n_+$ , and in that case :

$$\partial_t S = -\partial_i (\psi^+ M \alpha^i_+ \psi) + \partial_n (\psi^+ \gamma^0_+ \psi) \qquad \text{(A32)}$$

If $V^m$ is compact, then after integration on $V^m$ , the second term disappears, and it remains the divergence of the usual probability current.

If there are several coordinates such that $\eta^{aa} = +1$ , other than $a = 0$ , for instance $a = n_0$ and $a = n_1$ , one can chose : $M \sim \gamma_+^{n_0} \gamma_+^{n_1}$ . In this case , the relations :

$$\alpha_+^i M = M \alpha_+^i \qquad \text{and :} \qquad \gamma_+^0 \gamma_+^n M = -M \gamma_+^0 \gamma_+^n \tag{A33}$$

are still true , and (A32) keeps its form , and $S$ is conserved at the « macroscopic » level .

# Appendix B. Notations and basic geometrical equations.

The space-time coordinates of a point $x$ are labelled with Greek letters $\alpha, \beta, \gamma$ ... : $\{x^\alpha\}$ , $0 \le \alpha, \beta, \gamma, ... < n+m$ . The vectors of the local natural frame are written : $\overrightarrow{e_\alpha}, \overrightarrow{e_\beta}$ , ... . When tensors are expressed with respect to local orthonormal frames, they are labelled with Latin letters : $a, b, c$ ... . The orthonormal local frame basis vectors are called $\overrightarrow{h_a}$ , and we set : $\overrightarrow{h_a} = h_a^\alpha \overrightarrow{e_\alpha}$ . The metric tensor is $g_{\alpha\beta}$ and $g^{\alpha\beta}$ its inverse. In the case of local orthonormal frames, the metric tensor is written : $\eta_{ab}$ .

The commutator of the vectors $\{\overrightarrow{h_a}\}$ is :

$$[h_a, h_b]^\gamma = h_a^\alpha \partial_\alpha h_b^\gamma - h_b^\alpha \partial_\alpha h_a^\gamma = C^c_{.ab} h_c^\gamma \tag{B.1}$$

In the neighborhood of a given point, the local coordinates, with respect to the local orthonormal frame attached to this point, are given by the 1-forms : $\omega^a = h_\alpha^a dx^\alpha$ , which satisfy the structure equations : $\qquad d\omega^a + \omega^a_{.b} \wedge \omega^b = \Sigma^a \qquad$ (B.2)

where : $\omega^a_{.b} = \omega^a_{.b\gamma} dx^\gamma$ are the connexion 1-forms and $\Sigma^a$ are the torsion 2-forms. We shall also write: $\omega^a_{.b} = \omega^a_{.bc} \omega^c \leftrightarrow \omega^a_{.bc} = \omega^a_{.b\gamma} h_c^\gamma$ . The connexion 1-forms are related to the connexion coefficients by :

$$\omega^a_{.b\gamma} = \Gamma^\alpha_{.\beta\gamma} h_\alpha^a h_b^\beta + h_\delta^a \partial_\gamma h_b^\delta \tag{B.3}$$

The connexion coefficients are the sum of two terms :

$$\Gamma^\alpha_{.\beta\gamma} = \tilde{\Gamma}^\alpha_{.\beta\gamma} + \bar{S}^\alpha_{.\beta\gamma} \tag{B.4}$$

where the first term is the Christoffel symbol and the second is the contorsion tensor. The contorsion is anti symmetric with respect to the two first indices : $\bar{S}_{\alpha\beta\gamma} + \bar{S}_{\beta\alpha\gamma} = 0$ . The torsion tensor is :

$$S^\alpha_{.\beta\gamma} = \frac{1}{2}(\Gamma^\alpha_{.\beta\gamma} - \Gamma^\alpha_{.\gamma\beta}) = \frac{1}{2}(\bar{S}^\alpha_{.\beta\gamma} - \bar{S}^\alpha_{.\gamma\beta}) \tag{B.5a}$$

and inversely :

$$\bar{S}^\alpha_{.\beta\gamma} = S^\alpha_{.\beta\gamma} - S_{\beta.\gamma}^{\ \ \alpha} - S_{\gamma.\beta}^{\ \ \alpha} \quad ; \quad S_{\beta.\gamma}^{\ \ \alpha} = g^{\alpha\delta} S_{\beta\delta\gamma} \tag{B.5b}$$

The torsion 2-forms are : $\qquad \Sigma^a = \Sigma^a_{.bc} \omega^b \wedge \omega^c = -h_\alpha^a S^\alpha_{.\beta\gamma} dx^\beta \wedge dx^\gamma \qquad$ (B.6)

Using (B.3) we set :

$$\tilde{\Gamma}^a_{.b\gamma} = \tilde{\Gamma}^\alpha_{.\beta\gamma} h_\alpha^a h_b^\beta + h_\delta^a \partial_\gamma h_b^\delta \tag{B.7}$$

The curvature 2-forms are defined by :

$$\Omega^a_{.b} = d\omega^a_{.b} + \omega^a_{.c} \wedge \omega^c_{.b} = R^a_{.bcd} \omega^c \wedge \omega^d \tag{B.8}$$